Utrecht University
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# Reconstructive Geometry in certain Triangulated Categories

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*Dedicated to my parents Ingrid and Peter, who accompanied me with their love and support for the past six years that come to an end with the completion of this document.*

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CHAPTER 0

# Preface

Homological algebra has become a very important tool in many mathematical areas. In algebraic geometry, its language has contributed to a good understanding of many complex phenomena, like for example Serre duality. Coherent cohomology theory replaces the study of an algebraic variety $X$ (or more generally a noetherian scheme) by the study of derived functors on $\mathrm{Coh}(X)$, the category of coherent sheaves on $X$. Its objects are often easier understood than $X$ itself and in this sense, this approach provides a simplification. A more natural framework for the study of derived functors is $D^b(X)$, the bounded derived category of coherent sheaves on $X$ which is obtained from $\mathrm{Coh}(X)$ by a general procedure which can be applied to any "suitable" (abelian) category. Its objects are chain complexes of objects from $\mathrm{Coh}(X)$. When passing from $\mathrm{Coh}(X)$ to $D^b(X)$, we apply another simplification in the sense that two non-isomorphic chain complexes with objects in $\mathrm{Coh}(X)$ may become isomorphic as objects in $D^b(X)$ (in fact, any exact sequence becomes isomorphic to zero in $D^b(X)$!). Thus, the first part of this thesis comes as a small surprise: even though $D^b(X)$ could be considered a "simpler" mathematical object than $X$ itself, it still determines $X$ up to isomorphism, if we place a (not too strong) simplicity condition on $X$. This is Bondal and Orlov's theorem, for which we give a detailed proof in the first part of this thesis, as, in some sense, it forms the base for our further work: from the theorem's point of view, it seems a natural question if we can do algebraic geometry by forgetting about the varieties and schemes involved and just looking at categories that are of the same "type" as $D^b(X)$. A possible framework for these considerations is provided by tensor-triangulated categories. In the second part of this thesis, we try to define analogies of geometrical constructions for these categories (Chow groups, Hilbert polynomials, products) and we prove that our constructions recover the original geometrical ones, in the case that our categories come from varieties (or suitable schemes). This takes place in chapter 3 and 4 of this document and the focus lies on the Chow group of a variety, which is a basic tool for dealing with subvarieties of the variety and for intersection theory.

This thesis started as an attempt to thoroughly understand the theorem by A. Bondal and D. Orlov mentioned above, which states that a smooth algebraic variety with (anti-)ample canonical bundle is determined up to isomorphism by its derived category of coherent sheaves as a graded category. The proofs given in Bondal and Orlov's original paper and in Huybrechts' account turned out to be interesting itself: we can find numerous attempts there to "geometrize" the derived categories involved, i.e. attempts to carry over geometric concepts to a category theoretic setting. Therefore, it seemed natural to ask how much geometric structure derived categories - or more generally triangulated categories - were carrying in general. A partial answer to this question is given by the work of Hopkins, Neeman and Thomason, who proved that closed subvarieties of an algebraic variety correspond bijectively to certain thick subcategories of $D^{\mathrm{perf}}(X)$, the derived category of perfect complexes on $X$. From this starting point, Balmer succeeds to associate to every tensor-triangulated category $K$ a locally ringed space $\mathrm{Spec}(K)$ that is isomorphic to $X$ in case that $K = D^{\mathrm{perf}}(X)$ for some algebraic variety $X$. In the light of these results we therefore try to define the Chow group of an arbitrary tensor-triangulated category that reconstructs the Chow group of $X$ in case that $K = D^{\mathrm{perf}}(X)$ for some algebraic variety $X$. We are even able to define a partial intersection product on certain subcategories of $K$. In general, our constructions depend very much on Balmer's ringed space construction, and therefore the amount of analogies we can find between (algebraic) geometry and tensor-triangulated categories is somewhat constrained by the amount of theory we can generalize from schemes to arbitrary locally ringed spaces.

The document at hand has the following structure: the first part (chapters 1 and 2) is dedicated to the derived category of coherent sheaves on a smooth projective variety with ample canonical bundle. This means that we will give a (sometimes very) brief review of most of the geometry and homological algebra needed and then give a proof of Bondal and Orlov's theorem following Huybrechts' account. In the second part (chapters 3 and 4) we give an overview of the theory established by Balmer that we need and we present the following new results:

- We define Chow groups for certain tensor-triangulated categories.
- We give an outlook to a possible intersection theory defined on these categories.
- We define Hilbert functions, the arithmetic genus and a "geometric product" of certain tensor-triangulated categories.

This new part of the theory gives rise to some interesting and (as of now) unresolved problems. Specifically, we want to mention conjecture 4.5 from chapter 3 and its analogue for intersection multiplicities. In the same context, it would be desirable to have an analogue of Chow's moving lemma that works for the Chow group of a tensor-triangulated category $K$ as defined in chapter 3, as this could ultimately lead to a full intersection theory on $K$.

CHAPTER 1

# Preliminaries

The reader will need a good understanding of basic modern algebraic geometry, i.e. he should be familiar with the basic theory of schemes and the associated commutative algebra. Chapter II of [**Har77**] provides a good background and throughout the text, we will try to stick to the notation introduced there. Furthermore, we require a basic knowledge on category theory and sheaf cohomology, a general acquaintance with homological algebra will be useful as well. In the following we will first look at some notions of algebraic geometry that are used less frequently in [**Har77**] and recall some facts that are central for our work. Then, we will give a short overview of derived and triangulated categories, which are the central objects we study in this thesis. Finally, we will recall some facts from homological algebra we need and have a closer look at some properties of the derived category of coherent sheaves on a smooth projective variety. As a last reminder, we will very quickly go over spectral sequences as we need them several times when proving Bondal and Orlov's theorem.

## 1. Algebraic geometry recap

We begin with some very basic notions that we will use frequently throughout the text. We already assume some familiarity with these concepts, so we will not go into details everywhere.

**Definition 1.1.** *Let $X$ be a topological space. A non-empty, closed subset $Y \subset X$ is called* irreducible *if it is not the union of two proper closed subsets of $X$. If $X$ is irreducible, the* dimension of $X$ *is defined as*

$$dim(X) = \max_n \{\textit{Chains of irreducible subspaces } X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_n = X\}$$

**Definition 1.2.** *Let $X$ be a topological space, $A, B$ sheaves of abelian groups on $X$. Then we define the sheaf of local homomorphisms $\mathcal{H}om(A, B)$ on $X$ by*

$$U \mapsto Hom(A|_U, B|_U)$$

**Remark:** It is clear that $\mathcal{H}om(A, B)$ is a presheaf, with restriction maps the natural restriction of homomorphisms of sheaves. It is also a sheaf: let $U \subset X$ be open and $\{V_i\}$ be an open covering of $U$. First assume that we're given $s \in \mathcal{H}om(A, B)(U) = \mathrm{Hom}(A|_U, B|_U)$ such that $s|_{V_i} = 0$ for all $i$, i.e. $s \circ \rho_i^A = 0$, where $\rho_i^A$ is the restriction map $A(U) \longrightarrow A(V_i)$. As $s$ is a morphism of sheaves, this means that we also have $\rho_i^B \circ s = 0$ for all $i$. Now let $a \in A(U)$ be arbitrary and consider $s(a) \in B(U)$. We have that $\rho_i^B(s(a)) = 0$ for all $i$, which means that $s(a) = 0$ as $B$ is a sheaf. This proves that $s = 0$. Next, assume we are given elements $s_i \in \mathrm{Hom}(A|_{V_i}, B|_{V_i})$ such that $s_i|_{V_j \cap V_i} = s_j|_{V_j \cap V_i}$ for all $i, j$. Now we define a map $s : A(U) \longrightarrow B(U)$ in the following way: let $a \in A(U)$ then define $b_i := s_i \circ \rho_i^A(a)$. Then for each $V_i$, we get a $b_i \in V_i$ and by definition of the $s_i$ and the fact that they commute with the restriction maps we must have $b_i|_{V_j \cap V_i} = b_j|_{V_j \cap V_i}$ for all $i, j$. As $B$ is a sheaf, there is a unique element $b \in B$ such that $b|_{V_i} = b_i$. If we set $s(a) = b$ we can check that this gives a well-defined homomorphism of sheaves and we see that $\mathcal{H}om(A, B)$ is indeed a sheaf. It is easy to see that if we replace the sheaves of abelian groups by sheaves of modules, rings etc. the sheaf of local homomorphism becomes a sheaf in the corresponding category.

A definition which will be useful when working with points of a scheme is the following:

**Definition 1.3.** *Let $X$ be a topological space, $P \in X$ and $A$ an abelian group. The skyscraper sheaf $i_P(A)$ of $A$ at $P$ on $X$ is given by the assignment*

$$U \mapsto \begin{cases} A & \text{if } P \in U \\ 0 & \text{if } P \notin U \end{cases}$$

This is a sheaf as well: it is clear that $i_P(A)$ is a presheaf with restriction homomorphisms either the identity map or the zero map. It is also a sheaf: let $U \subset X$ be open and $\{V_i\}_{i \in I}$ an open covering of $U$, $s \in i_P(A)(U)$. Assume that $s \neq 0$, i.e. $P \in U$. Then for some $j$, we have $P \in V_j$ and therefore $s|_{V_j} \neq 0$. This implies that if $s|_{V_i} = 0$ for all $i \in I$ then we must have $s = 0$. Furthermore, if we have $U \subset X$ open, $\{V_i\}$ an open covering of $U$ and elements $s_i \in i_P(A)(V_i)$ for each $i$, with the property that for each $i, j$ we have $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then we need to show that there is an element $s \in i_P(A)(U)$ such that $s|_{V_i} = s_i$ for all $i$. But clearly, the conditions on the intersections, together with the fact that the restriction maps are either the identity map or the zero map, imply that $s_i = s_j = s$ for all $j$ and thus if we pick $s = s_i$ for some $i$, we get our desired element.

Skyscraper sheaves are important for the first part of this thesis, as they represent points on the projective variety $X$ in the derived category $D^b(X)$. One type of skyscraper sheaves that will be used extensively is the next one:

**Definition 1.4.** *Let $(X, \mathcal{O}_X)$ be a scheme and $x \in X$ a closed point. The sheaf $k(x)$ is the skyscraper sheaf $i_x(\mathcal{O}_{X,x}/\mathfrak{m}_x)$, where $\mathcal{O}_{X,x}$ is the local ring at $x$ and $\mathfrak{m}_x$ is its maximal ideal.*

**Remark 1:** Most of the time we will be working with schemes of finite type over an algebraically closed field $k$, and thus we have $\mathcal{O}_{X,x}/\mathfrak{m}_x = k$ for all closed points $x \in X$. Indeed, in this case every point $x \in X$ is contained in an open $\mathrm{Spec}(B)$, where $B$ is a finitely generated $k$-algebra and for $x$ closed, $\mathcal{O}_{X,x} \cong B_{(m)}$, where $m \subset B$ is the maximal ideal corresponding to $x$. We have $B_{(m)}/\mathfrak{m} \cong B/m$ which is a finite-dimensional $k$-vector space due to Hilbert's Nullstellensatz. As $k$ is algebraically closed, this implies the claim.

**Remark 2:** In the following, we will always consider $k(x)$ as a sheaf of $\mathcal{O}_X$-modules. This works in the following way: if $x \notin U \subset X$, then $k(x)(U) = 0$, so it is an $\mathcal{O}_X(U)$-module in the trivial sense. If $x \in U$, $s \in \mathcal{O}_X(U)$ and $a \in k(x)(U) = \mathcal{O}_{X,x}/\mathfrak{m}_x$, define $s \cdot a := \pi_x(s_x) \cdot a$, where $s_x$ is the stalk of $s$ at $x$ and $\pi_x : \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x$ is the natural projection map.

We will prove several more statements for skyscraper sheaves in the following:

**Proposition 1.5.** *Let $X$ be a topological space and assume $x \neq y \in X$ are two distinct closed points. And let $i_x(A)$ be a skyscraper sheaf in $x$ and $i_y(B)$ be a skyscraper sheaf in $y$. Then $Hom(i_x(A), i_y(B)) = 0$.*

PROOF. Consider the sheaf of local homomorphisms $\mathcal{H}om(i_x(A), i_y(B))$, then

$$\mathrm{Hom}(i_x(A), i_y(B)) = \Gamma(X, \mathcal{H}om(i_x(A), i_y(B)))$$

Define the open subsets $U = X \setminus \{x\}$ and $V = X \setminus \{y\}$. Then we have

$$\mathcal{H}om(i_x(A), i_y(B))|_U = 0$$

and

$$\mathcal{H}om(i_x(A), i_y(B))|_V = 0$$

and thus for any $f \in \Gamma(X, \mathcal{H}om(i_x(A), i_y(B))) = \mathrm{Hom}(i_x(A), i_y(B))$ we have $f|_U = 0$ and $f_V = 0$. As $U \cup V = X$ and $\mathcal{H}om(i_x(A), i_y(B))$ is a sheaf, this means that $f = 0$. □

A nice description for the global sections of a sheaf of $\mathcal{O}_X$-modules is given by the following lemma:

**Lemma 1.6.** *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on a scheme $X$. Then $\Gamma(X, \mathcal{F}) = Hom(\mathcal{O}_X, \mathcal{F})$.*

PROOF. First, assume that we have a homomorphism of $\mathcal{O}_X$-modules $f : \mathcal{O}_X \longrightarrow \mathcal{F}$. Then for each $U \subset X$, $f(U)$ is completely determined by the image of $1 \in \mathcal{O}_X(U)$ in $\mathcal{F}$: indeed, for all $a \in \mathcal{O}_X(U)$, we have $f(U)(a) = af(U)(1)$. Thus, we get a morphism of $\mathcal{O}_X(X)$-modules $\phi : \mathrm{Hom}(\mathcal{O}_X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F})$ by mapping $f \mapsto f(X)(1)$. On the other hand, assume that we are given some $a \in \mathcal{F}(X)$. Then we can define a morphism of $\mathcal{O}_X(X)$-modules $\psi' : \mathcal{F}(X) \longrightarrow \mathrm{Hom}_{\mathcal{O}_X(X)}(\mathcal{O}_X(X), \mathcal{F}(X))$ by putting $a \mapsto (f' : \mathcal{O}_X(X) \longrightarrow \mathcal{F}(X), f'(1) = a)$. Notice that $f'$ extends to a unique morphism of $\mathcal{O}_X$-modules $f : \mathcal{O}_X \longrightarrow \mathcal{F}$ in the following way: for each $U \subset X$, $f(U)$ is determined by $f(U)(1)$, but we have that $f(U)(1) = f(U)(\rho_U^X(1)) = \rho_U^X(f(X)(1))$, where the first equality follows from the fact that the restriction maps of $\mathcal{O}_X$ are ring-homomorphisms which send 1 to 1. In this way $\psi'$ induces a map of $\mathcal{O}_X(X)$-modules $\psi : \Gamma(X, \mathcal{F}) \longrightarrow \mathrm{Hom}(\mathcal{O}_X, \mathcal{F})$ which is clearly inverse to $\phi$. Thus, $\Gamma(X, \mathcal{F})$ and $\mathrm{Hom}(\mathcal{O}_X, \mathcal{F})$ are isomorphic as $\mathcal{O}_X(X)$-modules. □

The next definition gives a reminder of the notion of a coherent $\mathcal{O}_X$-module on a scheme $X$. For the $\sim$-construction, see [**Har77**, p.111]

**Definition 1.7.** *Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $F$ is called* quasi-coherent *if $X$ can be covered by finitely many open affines Spec $A_i =: U_i$ such that $F|_{U_i} = \widetilde{M_i}$, where the $M_i$ are $A_i$-modules. If the $M_i$ are finitely generated, then $F$ is called* coherent.

A simple example of a coherent sheaf is the following:

**Proposition 1.8.** *Let $X$ be a noetherian scheme and $x \in X$ be a closed point. Then the skyscraper sheaf $k(x)$ is a coherent sheaf of $\mathcal{O}_X$-modules.*

Proof. The coherence of $k(x)$ follows from [**Har77**, III.8.8.1] as the inclusion $i : \{x\} \longrightarrow X$ is a closed immersion of noetherian schemes, which is a proper morphism by [**Har77**, II.4.8]: indeed, we can view $k(x)$ as the structure sheaf $\mathcal{O}_{\{x\}}$ of the the singleton $\{x\}$, which is coherent. Then $k(x) = i_*(\mathcal{O}_{\{x\}})$. □

Coherent sheaves are used extensively for cohomology theory on schemes, and so we will later look at the derived category of coherent sheaves on a smooth projective variety. The main objects we will consider in the following are smooth (non-singular) algebraic varieties. Therefore we briefly recall the notion of the canonical bundle of a scheme (which is rather technical and we just state it for completeness' sake).

**Definition 1.9.** *Let $Y$ be a closed subscheme of a scheme $X$ and let $i : Y \longrightarrow X$ be the inclusion morphism. The* ideal sheaf of $Y$, $\mathcal{J}_Y$, *is the kernel of the morphism $i^\# : \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y$.*

**Definition 1.10.** *Let $X$ be a scheme over $k$ of dimension $n$, and $\Delta : X \longrightarrow X \times_k X$ be the diagonal morphism. Then $\Delta(X)$ is a closed subscheme of an open subset $W \subset X \times_k X$. Let $\mathcal{J}$ be the sheaf of ideals of $\Delta(X)$ in $W$. Then the* sheaf of relative differentials of $X$ over $k$ on $X$ *is defined as $\Omega_{X/k} := \Delta^*(\mathcal{J}/\mathcal{J}^2)$. If $X$ is non-singular, then we define the* canonical sheaf *or* canonical bundle $\omega_X$ *of $X$ as $\bigwedge^n \Omega_{X/k}$.*

**Remark:** For the pull-back construction $\Delta^*$, see [**Har77**, II.5]. We can also define the sheaf of differentials in the following way: cover $X$ with open affines $U_i := \text{Spec } A_i$, then define $\Omega_{X/k}|_{U_i} := \widetilde{\Omega_{A_i/k}}$ and glue all these sheaves together to obtain the sheaf $\Omega_{X/k}$. Here, $\Omega_{A_i/k}$ is the module of relative differential forms of $A_i$ over $k$ (cf. [**Har77**, II.8.9.2]).

$\Omega_{X/k}$ is especially well-behaved for non-singular varieties, i.e. for varieties over an algebraically closed field $k$ such that all its local rings are regular local rings.

**Theorem 1.11.** *Let $X$ be an irreducible separated scheme of finite type over an algebraically closed field $k$. Then $\Omega_{X/k}$ is a locally free sheaf of rank $n = \dim X$ if and only if $X$ is a non-singular variety over $k$.*

Proof. For a proof see [**Har77**, II.8.15] □

Thus, we have in particular:

**Corollary 1.12.** *Let $X$ be smooth projective variety over an algebraically closed field $k$. Then $\omega_X$ is an invertible sheaf (i.e. locally free of rank 1).*

Proof. Let $x \in X$ be a point. Then $\omega_{X,x} = \bigwedge^n(\Omega_{X/k})_x$. By the previous theorem this is a free $\mathcal{O}_{X,x}$-module of rank 1, which is equivalent to $\omega_X$ being invertible in an open neighbourhood of $x$. As $x$ was chosen arbitrarily, this proves the claim. □

**Definition 1.13.** *Let $X$ be a noetherian scheme. The support of a coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is defined as $supp(\mathcal{F}) := \{x \in X | \mathcal{F}_x \neq 0\}$. It is a closed subset of $X$.*

Proof. We want to prove that $\text{supp}(\mathcal{F})$ is closed. We can prove this locally (as a set that is closed in each part of an open covering is globally closed), i.e. reduce it to the situation, where $X = \text{Spec}(A)$ is affine. Then we can write $\mathcal{F} = \widetilde{F}$, where $F$ is some finitely generated $A$-module. Then we have that $\operatorname{supp}(\mathcal{F}) = V(\operatorname{Ann}(F))$: indeed, let $\mathfrak{p} \in V(\operatorname{Ann}(F))$ which means that $\mathfrak{p}$ is a prime ideal in $A$ that contains $\operatorname{Ann}(F)$. Now look at $F_\mathfrak{p}$ and assume that $F_\mathfrak{p} = 0$. This implies that for all generators $f_i \in F$ there are elements $a_i \in A \setminus \mathfrak{p}$ such that $a_i f_i = 0$. Then we have that $(\prod_i a_i)f = 0$ for all $f \in F$ and thus $(\prod_i a_i) \in \operatorname{Ann}(F)$ but also $(\prod_i a_i) \in A \setminus \mathfrak{p}$ which is a contradiction. For the other implication, take $\mathfrak{p} \in \operatorname{supp}(\mathcal{F})$, then $F_\mathfrak{p} \neq 0$. This implies that $\operatorname{Ann}(F) \subset \mathfrak{p}$. Indeed, assuming the opposite yields an element $a \notin \mathfrak{p}$ such that $aF = 0$ which implies that $F_\mathfrak{p} = 0$, a contradiction. This finishes the proof. □

The next lemma gives a nice characterization of sheaves with small support, and will be used several times for the proof of Bondal and Orlov's theorem:

**Lemma 1.14.** *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules with supp$(\mathcal{F}) = \{x\}$ a closed point in $X$. Then $\mathcal{F}$ is a subsheaf of the skyscraper sheaf $i_x(\mathcal{F}_x)$.*

Proof. We use the fact that as $\mathcal{F}$ is a sheaf, its sheafification $\mathcal{F}^+$ (cf. [**Har77**, II.1.2]) is isomorphic to $\mathcal{F}$ itself. Thus look at the sheafification $\mathcal{F}^+$, then $\mathcal{F}^+(U)$ is the set of functions $s : U \longrightarrow \coprod_{P \in U} \mathcal{F}_P$ subject to two conditions:

(1) for each $P \in U$, we have $s(P) \in \mathcal{F}_P$
(2) for each $P \in U$, there is a neighbourhood $V \ni P, V \subset U$ and $t \in \mathcal{F}(V)$ such that for all $Q \in V$ we have $t_Q = s(Q)$

Now we have two cases: if $x \notin U$, then $\coprod_{p \in U} \mathcal{F}_p$ is trivial and therefore the set of functions $s : U \longrightarrow \coprod_{p \in U} \mathcal{F}_p$ is trivial. If $x \in U$, then the first condition says that $s(P) \in \mathcal{F}_P$ for all $P \in U$. Thus as $\mathcal{F}_P = 0$ for $P \neq x$, $s$ is completely determined by the image $s(x) \in \mathcal{F}_x$. Therefore, we can identify $\mathcal{F}(U)$ with a subset of $S \subset \mathcal{F}_x$.

It remains to prove that $\mathcal{F}(V) = S$ for every open $V \subset X$ such that $x \in V$. Thus, let $V_1, V_2 \subset X$ be two opens containing $x$ and let $s \in \mathcal{F}(V_1)$. For each $P \in V_1$, we have $s(P) \in \mathcal{F}_P$ which means that we can extend $s$ to a function $\widetilde{s} : V_2 \longrightarrow \coprod_{P \in V_2} \mathcal{F}_P$ by setting $\widetilde{s}(P) = 0$ for $P \neq x$ and $\widetilde{s}(x) = s(x)$. In order to see that $\widetilde{s} \in \mathcal{F}(V_2)$, we need to check that for every point $P \in V_2$, there is a neighbourhood $W \ni P, W \subset V_2$ and $t \in \mathcal{F}(W)$ such that for all $Q \in W$ we have $t_Q = s(Q)$. For $P \neq x$, this is trivial, for $P = x$, we know that there is a neighbourhood $W' \subset V_1$ and a section $t' \in \mathcal{F}(W')$ with the desired property. Now take $W = W' \cap V_2$ and $t := t'|_{W' \cap V_2} \in \mathcal{F}(W)$ to get the neighbourhood and the section that we needed and see that $\widetilde{s} \in \mathcal{F}(V_2)$. By the very same argument we see that every element $t \in \mathcal{F}(V_2)$ gives rise to an element $\widetilde{t} \in \mathcal{F}(V_1)$. As we have seen before, elements of $\mathcal{F}(U)$ are determined completely by their value at $x$ for any open $U$ containing $x$ and thus we can check easily that the two processes described above are inverse to each other. This finishes the proof. □

The next two observations provide some category-theoretic background we need: first recall that for $X = \operatorname{Proj}(S)$ and $M$ a graded $S$-module, there is a associated sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ on $X$. (Note that there is some abuse of notation here: for an affine scheme $\operatorname{Spec}(A)$ and an $A$-module $N$, we denote the $\mathcal{O}_{\operatorname{Spec}(A)}$-module associated to $N$ also by $\widetilde{N}$. However, it should be clear from the context which construction is meant.)

**Theorem 1.15.** *Let $X = Proj(S)$ be a projective variety over $k$. Denote by $F$ the functor $X \mapsto \bigoplus_n H^0(X, \mathcal{O}_X(n))$ from projective schemes over $k$ to finitely generated $k$-algebras and by Proj the functor $A \mapsto Proj\ (A)$ from graded rings to projective schemes. Then $Proj\ \circ F = id$.*

PROOF. Recall that $\mathcal{O}_X(n) := \mathcal{O}_X \otimes \mathcal{O}(n)$ where $\mathcal{O}(n) = \widetilde{S(n)}$. Consider $S$ as an $S$-module, then have that $\widetilde{S} \cong \mathcal{O}_X$. Now by [**Har77**, II, Exercise 5.19] there is a natural graded homomorphism $S \longrightarrow \Gamma_*(\widetilde{S}) = \Gamma_*(\mathcal{O}_X) = \bigoplus_n H^0(X, \mathcal{O}_X(n))$ which is an isomorphism in all degrees $\geq k$ for some $k \in \mathbb{Z}$. By [**Har77**, II, Exercise 2.14] this implies that $X = \text{Proj}(S) = \text{Proj}(\bigoplus_n H^0(X, \mathcal{O}_X(n))) = \text{Proj}(F(X)) = \text{Proj} \circ F(X)$ which shows the claim. □

The next theorem is important as it implies that the category of coherent sheaves on a projective scheme $X$ over $k$ is Hom-finite and will eventually imply the same for the category $D^b(X)$ (which we will define later).

**Theorem 1.16.** *Let $X$ be a projective scheme over $k$ and let $\mathcal{A}, \mathcal{B}$ be two coherent $\mathcal{O}_X$-modules. Then $Hom(\mathcal{A}, \mathcal{B})$ is a finite-dimensional $k$-vector space.*

PROOF. Consider the sheaf of local homomorphisms $H := \mathcal{H}om(\mathcal{A}, \mathcal{B})$. We can cover $X$ with open affines $\text{Spec}(B_i)$ such that $\mathcal{B}|_{\text{Spec}(B_i)} = \widetilde{M_i}$ for some finitely generated $B_i$-module $M_i$. Then by [**Har77**, II.5.4], we have that $\mathcal{A}|_{\text{Spec}(B_i)} = \widetilde{N_i}$ for some finitely generated $B_i$-module $N_i$. Thus we have that $H(\text{Spec}(B_i)) = \text{Hom}(\mathcal{A}|_{\text{Spec}(B_i)}, \mathcal{B}|_{\text{Spec}(B_i)}) = \text{Hom}(\widetilde{N_i}, \widetilde{M_i}) = \text{Hom}(N_i, M_i)$ by [**Har77**, II.5.5]. This also shows that $H|_{\text{Spec}(B_i)} = \widetilde{\text{Hom}(N_i, M_i)}$. But as $N_i, M_i$ are finitely generated, $\text{Hom}(N_i, M_i)$ is a finitely generated $B_i$-module. This proves that $H$ is a coherent sheaf on $X$. Now, by [**Har77**, II.5.19] $\text{Hom}(\mathcal{A}, \mathcal{B}) = \Gamma(X, H)$ is a finite-dimensional $k$-vector space. □

The following lemma helps us when dealing with Zariski tangent vectors of a variety in a point. It makes it possible to rephrase the conditions of a locally free sheaf separating points and tangent vectors in a convenient way (cf. Lemma 1.19).

**Definition 1.17.** *Let $X$ be a scheme over an algebraically closed field $k$. For any $x \in X$, let $\mathfrak{m}_x$ be the maximal ideal of the local ring at $x$. The* Zariski tangent space $T_x$ to $X$ at *$x$ is the dual of the $k$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$.*

**Lemma 1.18.** *Let $X$ be a smooth projective variety and $x \in X$. Then there is a bijection between closed subschemes $Z$ concentrated in $x$ with structure sheaf of the form $k[\epsilon]/(\epsilon^2)$ and the Zariski tangent vectors at $x$, i.e. elements of $T_x$.*

PROOF. (This is [**Har77**, Exercise II.2.8].) Let $Z$ be closed subscheme concentrated in $x$ with structure sheaf $k[\epsilon]/(\epsilon^2)$. Let $f^\# : \mathcal{O}_X \longrightarrow \mathcal{O}_Z$ be the surjection induced by the inclusion $Z \hookrightarrow X$. In particular, we have a surjective map $f_x^\# : \mathcal{O}_{X,x} \longrightarrow k[\epsilon]/(\epsilon^2)$. The maximal ideal of $k[\epsilon]/(\epsilon^2)$ is $(\epsilon)$ and as $f_x^\#$ is a ring homomorphism, $(f_x^\#)^{-1}((\epsilon))$ will be a maximal ideal in $\mathcal{O}_{X,x}$, which means that it must be equal to $\mathfrak{m}_x$. This also shows that $\mathfrak{m}_x^2 \subset \ker(f_x^\#)$ and thus $f_x^\#$ factors as $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x^2 \twoheadrightarrow k[\epsilon]/(\epsilon^2)$ and we can restrict it to a map $\mathfrak{m}_x/\mathfrak{m}_x^2 \twoheadrightarrow k \cdot \epsilon \cong k$ which is obviously an element of $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$.

Thus, we see that $Z$, together with its embedding in $X$, determines a tangent vector at $x$.

On the other hand, let $f \in (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ be a tangent vector at $x \in X$. Define a closed subscheme structure on $x$ in the following way: set $\mathcal{O}_Z := k[\epsilon]/(\epsilon^2)$ and denote by $f$ the inclusion of the point $x \in X$. We need to give a surjective homomorphism of sheaves $f^\# : \mathcal{O}_X \longrightarrow i_*(\mathcal{O}_Z)$. This is equivalent to giving a surjective ring homomorphism $\mathcal{O}_{X,x} \longrightarrow k[\epsilon]/(\epsilon^2)$. As we've seen before such a morphism will factor as $\mathcal{O}_{X,x} \twoheadrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x^2 \longrightarrow k[\epsilon]/(\epsilon^2)$. By assumption, we have a linear map $f : \mathcal{O}_{X,x}/\mathfrak{m}_x^2 \supset \mathfrak{m}_x/\mathfrak{m}_x^2 \longrightarrow k$. We will extend this map to a ring homomorphism $\hat{f} : \mathcal{O}_{X,x}/\mathfrak{m}_x^2 \longrightarrow k[\epsilon]/(\epsilon^2)$ in the following way: denote by $\pi : \mathcal{O}_{X,x}/\mathfrak{m}_x^2 \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x = k \subset \mathcal{O}_{X,x}$ the natural projection and notice that for all $a \in \mathcal{O}_{X,x}$ we have that $a - \pi(a) \in \mathfrak{m}_x/\mathfrak{m}_x^2$. Indeed, $\pi(a - \pi(a)) = \pi(a) - \pi^2(a) = \pi(a) - \pi(a) = 0$ and thus $a - \pi(a) \in \ker(\pi) = \mathfrak{m}_x/\mathfrak{m}_x^2$. Now define

$$\hat{f}(a) = \pi(a) + f(a - \pi(a))\epsilon$$

This is a ring homomorphism: additivity is clear, as both $\pi$ and $f$ are additive. Now, let $a, b \in \mathcal{O}_{X,x}$. Then

$$\begin{aligned}\hat{f}(ab) &= \pi(ab) + f(ab - \pi(ab))\epsilon \\ &= \pi(a)\pi(b) + f(ab - \pi(a)\pi(b))\epsilon\end{aligned}$$

and

$$\begin{aligned}\hat{f}(a) \cdot \hat{f}(b) &= (\pi(a) + f(a - \pi(a))\epsilon)(\pi(b) + f(b - \pi(b))\epsilon) \\ &= \pi(a)\pi(b) + \pi(a)f(b - \pi(b))\epsilon + \pi(b)f(a - \pi(a))\epsilon + f(a - \pi(a))f(b - \pi(b))\epsilon^2 \\ &= \pi(a)\pi(b) + \pi(a)f(b - \pi(b))\epsilon + \pi(b)f(a - \pi(a))\epsilon \text{ as } \epsilon^2 = 0\end{aligned}$$

which means

$$\begin{aligned}\hat{f}(ab) - \hat{f}(a) \cdot \hat{f}(b) &= f(ab - \pi(a)\pi(b))\epsilon - \pi(a)f(b - \pi(b))\epsilon - \pi(b)f(a - \pi(a))\epsilon \\ &= f(ab - \pi(a)\pi(b))\epsilon - f(\pi(a)(b - \pi(b)))\epsilon - f(\pi(b)(a - \pi(a)))\epsilon \\ &= f(ab - \pi(a)\pi(b))\epsilon - f(\pi(a)(b - \pi(b)) + \pi(b)(a - \pi(a)))\epsilon \\ &= f(ab - \pi(a)\pi(b))\epsilon - f(\pi(a)b + \pi(b)a - 2\pi(a)\pi(b))\epsilon \\ &= f(ab - \pi(a)\pi(b) - \pi(a)b - \pi(b)a + 2\pi(a)\pi(b))\epsilon \\ &= f(ab - \pi(a)b - \pi(b)a + \pi(a)\pi(b))\epsilon \\ &= f((\pi(a) - a)(\pi(b) - b))\epsilon \\ &= f(0)\epsilon = 0 \text{ as } (\pi(a) - a)(\pi(b) - b) \in (\mathfrak{m}_x/\mathfrak{m}_x^2)^2 = 0\end{aligned}$$

As the two process described above are clearly inverse to each other, this finishes the proof. □

As a last statement, we need a little technicality concerning very ample line bundles, which we will use for the proof of Bondal and Orlovs's theorem. Essentially, this is a reformulation of [**Har77**, II, Proposition 7.3] in terms of surjectivity conditions on the restriction maps.

**Lemma 1.19.** *Let $X$ be projective scheme over an algebraically closed field and let $\mathcal{F}$ be an invertible sheaf on $X$. Then $\mathcal{F}$ is very ample if and only if*

*(a) for any two distinct closed points $P, Q \in X$, the restriction map*

$$\mathcal{F} \longrightarrow \mathcal{F}_P/\mathfrak{m}_P\mathcal{F}_P \oplus \mathcal{F}_Q/\mathfrak{m}_Q\mathcal{F}_Q \cong k(P) \oplus k(Q)$$

*induces a surjection $\varphi : \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, k(P) \oplus k(Q))$.*

*(b) Let $P \in X$, $y \in (\mathfrak{m}_P/\mathfrak{m}_P^2)^\vee$ be any tangent vector. Then the map $\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, i_*(\mathcal{O}_{P,y}))$ induced by the restriction $\mathcal{F} \longrightarrow \mathcal{F}_P$ is surjective. Here $\mathcal{O}_{P,y}$ is the structure sheaf of the closed subscheme concentrated in $P$ associated to the tangent vector $y$.*

Proof. Let $\mathcal{F}$ be very ample, i.e. $\mathcal{F} \cong i^*(\mathcal{O}(1))$. We will make use of the fact that $\mathcal{F}$ is very ample if and only if it separates points and tangent vectors (cf. [**Har77**, II, Proposition 7.3]). As $F$ separates points, we know that for any two distinct close points $P, Q \in X$ we have a global section $s$ such that $s_P \in \mathfrak{m}_P\mathcal{F}_P$ but $s_Q \notin \mathfrak{m}_Q\mathcal{F}_Q$, or vice versa. Next, let $\overline{(a, b)} \in \Gamma(X, \mathcal{F}_P/\mathfrak{m}_P\mathcal{F}_P \oplus \mathcal{F}_Q/\mathfrak{m}_Q\mathcal{F}_Q)$ and pick $s_1 \in \Gamma(X, \mathcal{F})$ such that $\overline{(s_1)_P} = a$ and $\overline{(s_1)_Q} = 0$. We can pick $s_1$ like this because we can pick $s \in \Gamma(X, \mathcal{F})$ such that $s_P \notin \mathfrak{m}_P\mathcal{F}_P, s_Q \in \mathfrak{m}_Q\mathcal{F}_Q$ by assumption. Then $0 \neq \overline{s_P} \in k$ and thus we put $s_1 = (a/\overline{s_P})s$ to get our desired element. Similarly, we pick $s_2 \in \Gamma(X, \mathcal{F})$ such that $\overline{(s_2)_P} = 0$ and $\overline{(s_2)_Q} = b$. Then it is clear that $\varphi(s_1 + s_2) = (a, b)$ and thus $\varphi$ is surjective.

Next we want to prove that $\mathcal{F}$ being very ample implies property (b). As $\mathcal{F}$ separates tangent vectors, we know that for each point $P \in X$, the set $\{s_P | s \in \Gamma(X, \mathcal{F}), s_P \in \mathfrak{m}_P\}$ spans the vector space $\mathfrak{m}_P\mathcal{F}_P/\mathfrak{m}_P^2\mathcal{F}_P \cong \mathfrak{m}_P/\mathfrak{m}_P^2$. Now let $y \in (\mathfrak{m}_P/\mathfrak{m}_P^2)^\vee$ and denote by $(P, (\mathcal{O}_{P,y}))$ the corresponding closed subscheme of $X$. The map $\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, i_*(\mathcal{O}_{P,y}))$ is the just restriction $\rho : \Gamma(X, \mathcal{F}) \longrightarrow \mathcal{F}_P \cong \mathcal{O}_P$ followed by the projection $p : \mathcal{O}_P \longrightarrow \mathcal{O}_P/\mathfrak{m}_P^2$ and the map $\sigma : \mathcal{O}_P/\mathfrak{m}_P^2 \longrightarrow k[\epsilon]/\epsilon^2$ which is given by the assignment

$$a \mapsto \pi(a) + y(a - \pi(a))\epsilon$$

(for the notation also see the proof of lemma1.18). As $\mathcal{F}$ separates tangent vectors, we know that $p \circ \rho$ maps surjectively on $\mathfrak{m}_P/\mathfrak{m}_P^2 \subset \mathcal{O}_P/\mathfrak{m}_P^2$, which also proves that the composition $\sigma \circ p \circ \rho$ is surjective.

On the other hand, let $\mathcal{F}$ be an invertible sheaf on $X$ such that the map $\varphi : \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, k(P) \oplus k(Q))$ is surjective for any two closed points $P, Q \in X$. This means that we can pick elements $s_1, s_2 \in \Gamma(X, \mathcal{F})$ such that $\varphi(s_1) = (a, 0)$ for any $0 \neq a \in k$ and $\varphi(s_2) = (0, b)$ for any $0 \neq b \in k$. This precisely means that for any two distinct close points $P, Q \in X$ we have a global section $s$ such that $s_P \in \mathfrak{m}_P\mathcal{F}_P$ but $s_Q \notin \mathfrak{m}_Q\mathcal{F}_Q$, or vice versa. This implies that any invertible sheaf that satisfies property (a) separates points.

As the last missing step, we want to show that an invertible sheaf $\mathcal{F}$ on $X$ that satisfies property (b) separates tangent vectors. Thus, let $P \in X$, then we need to show that the set $\{s_P | s \in \Gamma(X, \mathcal{F}), s_P \in \mathfrak{m}_P\}$ spans the vector space $\mathfrak{m}_P\mathcal{F}_P/\mathfrak{m}_P^2\mathcal{F}_P \cong \mathfrak{m}_P/\mathfrak{m}_P^2$. In order to prove this, assume the contrary, i.e. assume there is some $v \in \mathfrak{m}_P/\mathfrak{m}_P^2$ such that $v \notin \operatorname{im}(p \circ \sigma)$. Any $v \in \mathfrak{m}_P/\mathfrak{m}_P^2$ naturally gives rise to an element $\widetilde{v} \in (\mathfrak{m}_P/\mathfrak{m}_P^2)^\vee$ and we can check that for the subscheme associated to $\widetilde{v}$, the map $\sigma \circ p \circ \rho$ is not surjective, which gives a contradiction. This concludes the proof. □

## 2. Category-theoretic setting

In this section, we take a closer look at the category-theoretic setting we will be working in. Specifically, we will introduce the concepts of derived and triangulated categories, which will be central for our further considerations.

### 2.1. Abelian categories, chain complexes and the homotopy category.

**Definition 2.1.** *Let $\mathfrak{C}$ be a category. $\mathfrak{C}$ is called* abelian *if*

*(1) for each two objects $X, Y$ in $\mathfrak{C}$, $Hom_{\mathfrak{C}}(X, Y)$ has the structure of an abelian group and behaves linearly with respect to composition,*
*(2) $\mathfrak{C}$ has a* zero object *, i.e. an object $O$ such that $Hom(O, O)$ is the trivial group,*
*(3) finite sums and products exist and they are isomorphic,*
*(4) Every morphism $f \in Hom(A, B)$ admits a kernel and a cokernel and the natural map $Coim(f) \longrightarrow Im(f)$ is an isomorphism*

*A category that satisfies the first three axioms is called* additive. *An* additive functor *is a functor of additive categories that induces group-homomorphisms on the Hom-sets.*

**Definition 2.2.** *Let $k$ be a field. An additive category $\mathfrak{C}$ is called $k$-*linear*, if for every two objects $X, Y$ of $\mathfrak{C}$ we have that $Hom_{\mathfrak{C}}(X, Y)$ is a $k$-vector space and composition of maps is bilinear.*

**Example:** The most important example of an abelian category for us is $\text{Coh}(X)$, the category of coherent sheaves on an algebraic variety $X$ over a field $k$ (the proof is an immediate consequence of [**Har77**, Proposition II.5.7]). It is also $k$-linear as one easily checks. A more general example, which will prove useful later on is the category $\text{Mod}(X)$ of sheaves of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$ (cf. [**Har77**, Example III.1.0.6]).

**Definition 2.3.** *Let $\mathcal{A}$ be an abelian category. A* chain complex *with objects in $\mathcal{A}$ is a family of objects $B^\bullet = (B^i)_{i\in\mathbb{Z}}$ and maps $d^i : B^i \longrightarrow B^{i+1}, i \in \mathbb{Z}$ with the property that $d^{i+1} \circ d^i = 0$ (or equivalently $im(d^i) \subset ker(d^{i+1})$), called the* differentials *of $B^\bullet$. We will visualize a chain complex $B^\bullet$ in the following way:*

$$\ldots \longrightarrow B^{i-1} \longrightarrow B^i \longrightarrow B^{i+1} \longrightarrow \ldots$$

*For a chain complex $B^\bullet$ with objects in $A$ and $n \in \mathbb{Z}$, we define the $n$-*th cohomology object *of $B^\bullet$ as*

$$H^n(B^\bullet) := ker(d^n)/im(d^{n-1})$$

*Note that this is an element of $\mathcal{A}$ for all $n$, as $\mathcal{A}$ has kernels and cokernels. A chain complex $B^\bullet$ is called* bounded from the left *if there is an integer $j$ such that $B^i = 0$ for all $i < j$. A chain complex $B^\bullet$ is called* bounded from the right *if there is an integer $k$ such that $B^i = 0$ for all $i > k$. A chain complex is called* bounded *if it is both bounded from the left and bounded from the right.*

**Definition 2.4.** *Let $A^\bullet, B^\bullet$ be chain complexes with objects in $\mathcal{A}$. A* morphism of chain complexes *$A^\bullet \longrightarrow B^\bullet$ is a family of morphisms $f^i : A^i \longrightarrow B^i$ such that $d^i_B \circ f^i = f^{i+1} \circ d^i_A$ for all $i$. The morphism $f$ naturally induces morphisms $H^i(A^\bullet) \longrightarrow H^i(B^\bullet)$ for all $i$ and $f$ is called a* quasi-isomorphism *if these induced maps are all isomorphisms.*

**Definition 2.5.** *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a functor. $F$ is called*

- left exact, *if for any exact sequence* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *with objects in* $\mathcal{A}$*, the sequence* $0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$ *is exact.*
- right exact, *if for any exact sequence* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *with objects in* $\mathcal{A}$*, the sequence* $F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0$ *is exact.*

$F$ *is called* exact *if it is both left exact and right exact.*

**Definition 2.6.** *Let* $\mathcal{A}$ *be an abelian category.*

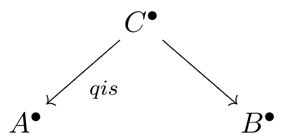
- *The* category of chain complexes in $\mathcal{A}$, denoted by $Ch(\mathcal{A})$ *is the category that has objects all chain complexes of objects in* $\mathcal{A}$ *and morphisms maps of chain complexes.*
- *The* homotopy category $K(\mathcal{A})$ *is the category that has the same objects as* $Ch(\mathcal{A})$ *and as morphisms maps of chain complexes modulo homotopy equivalence. This means that* $Hom_{K(\mathcal{A})}(X, Y) = Hom_{Ch(\mathcal{A})}(X, Y)/\sim$*, where* $f \sim g$ *if there are maps* $h^n : X^n \longrightarrow Y^{n-1}$ *such that* $f^n - g^n = d_Y^{n-1} \circ h^n + h^{n+1} \circ d_X^n$ *for all* $n \in \mathbb{Z}$*.*

**Remark:** Note that a morphisms $A^\bullet \longrightarrow B^\bullet$ induces maps $H^n(A^\bullet) \longrightarrow H^n(B^\bullet)$ for all $n$. We can easily check that this makes $H^n$ into a functor $Ch(\mathcal{A}) \longrightarrow \mathcal{A}$. It is also true that two homotopy-equivalent maps induce the same map on the cohomology objects and thus this gives functors $H^n : K(\mathcal{A}) \longrightarrow \mathcal{A}$.

**2.2. The derived category of an abelian category.** To every abelian category $\mathcal{A}$, we can associate the so-called derived category $D(\mathcal{A})$. We construct it in the following way:

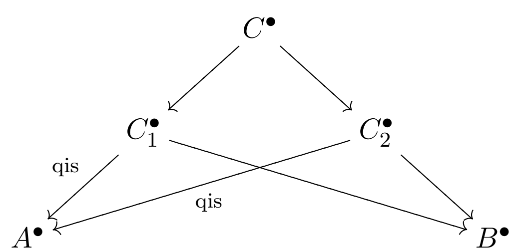
**Definition 2.7.** *Let* $\mathcal{A}$ *be an abelian category. Define the* derived category $D(\mathcal{A})$ *of* $\mathcal{A}$ *as follows:*

- *The objects of* $D(\mathcal{A})$ *are the objects of* $Ch(\mathcal{A})$
- *For* $A^\bullet, B^\bullet \in D(\mathcal{A})$ *the set of morphisms* $Hom_{D(\mathcal{A})}(A^\bullet, B^\bullet)$ *is given by equivalence classes of diagrams of the form*



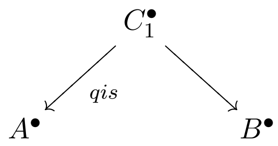

*where* $C^\bullet \longrightarrow A^\bullet$ *is a quasi-isomorphism. The equivalence relation and composition of maps are explained below.*

Two diagrams are considered equivalent if they are dominated by a third one in $K(\mathcal{A})$ in the following way:

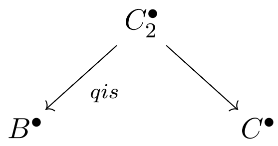


where the compositions $C^\bullet \longrightarrow C_1^\bullet \longrightarrow A^\bullet$ and $C^\bullet \longrightarrow C_2^\bullet \longrightarrow A^\bullet$ are quasi-isomorphisms and the diagram commutes in $K(\mathcal{A})$ i.e. up to homotopy equivalence.

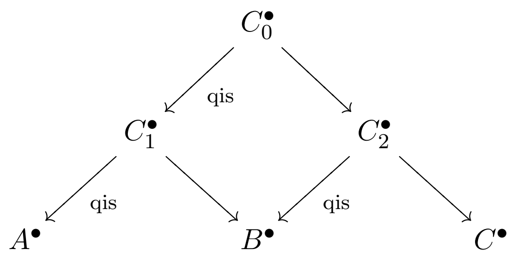
The composition of maps requires a bit of work (which we will not carry out explicitly): take two morphisms




and




then we want their composition to be given by a commutative (up to homotopy equivalence) diagram of the form




We will not give a proof here that these diagrams exist and that they're unique up to homotopy equivalence, but rather refer the reader to [**Huy06**, Chapter 2].

**Remark 1:** $D(\mathcal{A})$ is also characterized by the following universal property: let $i : Ch(\mathcal{A}) \longrightarrow D(\mathcal{A})$ be the canonical functor. Then $i$ sends quasi-isomorphisms to isomorphisms and if a functor $F : Ch(\mathcal{A}) \longrightarrow C$ sends quasi-isomorphisms to isomorphisms then there is a unique functor $G : D(\mathcal{A}) \longrightarrow C$ such that $F = G \circ i$. (cf. [**Huy06**, Theorem 2.10]).

**Remark 2:** One idea behind the derived category $D(A)$ is that we want to identify objects of $\mathcal{A}$ with their resolutions: let $\ldots \longrightarrow R^2 \longrightarrow R^1 \longrightarrow R^0 \xrightarrow{\epsilon} E$ be a resolution of an object $E \in \mathcal{A}$. If we consider the complex $\ldots \longrightarrow R^2 \longrightarrow R^1 \longrightarrow R^0 \longrightarrow 0 \longrightarrow \ldots$ as an object in $D(\mathcal{A})$ then we have a quasi-isomorphism from this complex to the complex $\ldots \longrightarrow 0 \longrightarrow E \longrightarrow 0 \longrightarrow \ldots$. But this means that there is also an inverse to this quasi-isomorphism in $D(\mathcal{A})$, which implies that both complexes are isomorphic in $D(\mathcal{A})$.

The following lemma tells us that cohomology also descends to the derived category and therefore provides a valuable tool for our further studies:

**Lemma 2.8.** *The cohomology functors $H^n : Ch(\mathcal{A}) \longrightarrow \mathcal{A}$ induce well-defined functors $H^n : D(\mathcal{A}) \longrightarrow \mathcal{A}$.*

Proof. cf. [**Huy06**, Chapter 2] □

A first useful application of cohomology is the following proposition.

**Proposition 2.9.** *There is an equivalence of categories between an abelian category $\mathcal{A}$ and the full subcategory of $D(\mathcal{A})$ consisting of $H^0$-objects (i.e. objects $P$ for which $H^i(P) \neq 0 \Leftrightarrow i = 0$).*

Proof. This follows from the fact that we have a quasi-isomorphism between a complex $A^\bullet$ with $H^i(A^\bullet) = 0$ for all $i > m$ and a complex $B^\bullet$ with $B^i = 0$ for all $i > m$ in the following way:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d^{m-2}} & A^{m-1} & \xrightarrow{d^{m-1}} & A^m & \xrightarrow{d^m} & A^{m+1} & \xrightarrow{d^{m+1}} & \cdots \\
 & & {\scriptstyle id}\uparrow & & {\scriptstyle i}\uparrow & & {\scriptstyle 0}\uparrow & & \\
\cdots & \xrightarrow{d^{m-2}} & A^{m-1} & \xrightarrow{d^{m-1}} & \ker(d^m) & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots
\end{array}$$

In the same way we construct a a quasi-isomorphism between a complex $A^\bullet$ with $H^i(A^\bullet) = 0$ for all $i < m$ and a complex $B^\bullet$ with $B^i = 0$ for all $i < m$:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d^{m-2}} & A^{m-1} & \xrightarrow{d^{m-1}} & A^m & \xrightarrow{d^m} & A^{m+1} & \xrightarrow{d^{m+1}} & \cdots \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle id} & & \\
\cdots & \xrightarrow{d^{m-2}} & 0 & \xrightarrow{d^{m-1}} & \mathrm{coker}(d^{m-1}) & \xrightarrow{d^m} & A^{m+1} & \xrightarrow{d^{m+1}} & \cdots
\end{array}$$

Now, if $C^\bullet$ is an $H^0$-object, then we can apply both procedures to show that in $D(\mathcal{A})$, we have $C^\bullet \cong H^0(C^\bullet)^\bullet_0$, where the latter is the complex that it equal to zero everywhere except at position 0, where it is $H^0(C^\bullet)$. Now it is clear that the functor $A \mapsto A^\bullet_0$ is essentially surjective, where $A^\bullet_0$ is the complex that is zero everywhere except at 0, where it is $A$. For the proof that it is fully faithful, we refer the reader to [**GM03**, Proposition III.5.2]. □

**Notation:** This proposition allows for some abuse of notation, as we can now view $H^0$-objects in $D(\mathcal{A})$ as objects of $\mathcal{A}$ and vice versa. Thus for $A \in \mathcal{A}$, we define $A[n] \in D(\mathcal{A})$ as the complex that is all zero except at position $n$, where it is $A$. If we leave out $[n]$ completely we mean $A[0]$.

Most of the time, we want to concentrate our efforts on bounded complexes. Therefore we make the following definition:

**Definition 2.10.** *Let $\mathcal{A}$ be an abelian category and define $Ch^*(\mathcal{A})$ for $* = +, -, b$ as the category of complexes $A^\bullet$ with $A^i = 0$ for $i \ll 0, i \gg 0$ or $|i| \gg 0$ respectively. Then by the same procedure as described above we obtain from $Ch^*(\mathcal{A})$ the category $D^*(\mathcal{A})$ and natural functors $D^*(\mathcal{A}) \longrightarrow D(\mathcal{A})$ by just forgetting the boundedness condition.*

Now we define the most important example of a bounded derived category for this thesis:

**Definition 2.11.** *Let $X$ be a noetherian scheme and consider the abelian category $Coh(X)$ of coherent sheaves on $X$. Then $D^b(X) := D^b(Coh(X))$ is defined as the bounded derived category of coherent sheaves on $X$.*

In general, bounded derived categories categories have a nice description

**Proposition 2.12.** *The natural functors $D^*(\mathcal{A}) \longrightarrow D(\mathcal{A})$ for $* = +, -, b$ define equivalences of $D^*(\mathcal{A})$ with the full subcategories of all complexes $A^\bullet \in D(\mathcal{A})$ with $H^i(A^\bullet) = 0$ for $i \ll 0, i \gg 0$ or $|i| \gg 0$ respectively.*

PROOF. cf. [**Huy06**, Proposition 2.30] □

**2.3. Triangulated categories.** Derived categories will play an important role in this thesis. A concept that captures their most important features is provided by the definition of a triangulated category. It is especially useful as it makes no reference to the underlying abelian categories.

**Definition 2.13.** *Let $\mathcal{D}$ be an additive category. The structure of a* triangulated category *on $\mathcal{D}$ is given by an additive equivalence*

$$T : \mathcal{D} \longrightarrow \mathcal{D},$$

*called the* shift functor *on $\mathcal{D}$ and a set of* distinguished triangles *of the form*

$$A \longrightarrow B \longrightarrow C \longrightarrow T(A)$$

*with $A, B, C \in \mathcal{D}$, which must satisfy a certain number of axioms.*

**Remark:** We will not specify the axioms that must be satisfied by the distinguished triangles as we will hardly need them. They can, for instance, be found in [**GM03**, Chapter IV.1]. One can think of the exact triangles as a generalization of short exact sequences. For a triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow T(A)$$

it is possible to prove from the axioms that the composition $A \longrightarrow C$ is zero, cf. [**Huy06**, p. 13]

**Notation:** It is, of course, possible to apply $T$ or its quasi-inverse a number of times to an object $A \in \mathcal{D}$. We will write $T^n A =: A[n]$ for all $n \in \mathbb{Z}$.

The next proposition gives us the most important example of a triangulated category for this thesis:

**Proposition 2.14.** *The derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$ is triangulated.*

PROOF. A proof can be found in [**GM03**, Chapter IV.2]. □

**Remark:** $D^b(X)$ is triangulated, with shift functor just the usual shifting of complexes in degree (i.e. $T(C^\bullet)$ is the complex with $T(C^\bullet)^i = C^{i+1}$ and differentials $d^i_{T(C^\bullet)} = -d^{i+1}_{C^\bullet}$). The set of distinguished triangles is defined via the mapping cone of a morphism of complexes (cf. [**Huy06**, Definition 2.23]).

Next, we need a replacement for the notion of an exact functor between abelian categories. Notice that many triangulated categories are not abelian. Therefore, kernels, cokernels etc. do not always exist, which, in general, makes it impossible to apply the usual definition of an exact functor, as we cannot even define an exact sequence.

**Definition 2.15.** *An additive functor*

$$F : \mathcal{D} \longrightarrow \mathcal{D}'$$

*between two triangulated categories is called* exact *if* *(i) There exists a functor isomorphism*

$$F \circ T_{\mathcal{D}} \cong T_{\mathcal{D}'} \circ F$$

*(ii) Any distinguished triangle*

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

*in $\mathcal{D}$ is mapped to a distinguished triangle*

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow F(A[1]) \cong F(A)[1]$$

We also need the following:

**Definition 2.16.** *A collection $\Omega$ of objects in a triangulated category $\mathcal{D}$ is called a* spanning class of $\mathcal{D}$ *if for all objects $B$ of $\mathcal{D}$ the following two conditions hold:*

*(1) If $Hom(A, B[i]) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$*
*(2) If $Hom(B[i], A) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$*

**Remark:** If $\mathcal{D}$ is $k$-linear and carries the additional structure of a Serre functor (cf. Definition 5.3), then the two conditions are equivalent: let $\mathrm{Hom}(A, B[i]) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$. Then we have that $0 = \mathrm{Hom}(A, B[i]) = \mathrm{Hom}(S(B[i]), A)^* = \mathrm{Hom}(S(B)[i], A)^*$ which implies that $\mathrm{Hom}(S(B)[i], A) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$. As $S$ is an auto-equivalence, this proves the claim.

An important feature of the derived category of an abelian category is the existence of the cohomology functors $H^n : D(\mathcal{A}) \longrightarrow \mathcal{A}$. This can be generalized to triangulated categories:

**Definition 2.17.** *Let $K$ be a triangulated category, $\mathcal{A}$ an abelian category and $H : K \longrightarrow \mathcal{A}$ be an additive functor. $H$ is a* cohomological functor *if for every distinguished triangle*

$$X \stackrel{u}{\longrightarrow} Y \stackrel{v}{\longrightarrow} Z \stackrel{w}{\longrightarrow} X[1]$$

*the sequence*

$$H(X) \stackrel{H(u)}{\longrightarrow} H(Y) \stackrel{H(v)}{\longrightarrow} H(Z)$$

*is exact.*

**Example:** Let $\mathcal{A}$ be an abelian category and consider the bounded derived category $D^b(\mathcal{A})$. The the standard 0-th cohomology functor $H^0 : D^b(\mathcal{A}) \longrightarrow \mathcal{A}$ is a cohomological functor (cf. [**GM03**, IV.1.6]). We obtain the functor $H^n$ for every $n$ by precomposing $H^0$ suitably many times with the shift functor.

## 3. Derived functors

Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F : \mathcal{A} \longrightarrow \mathcal{B}$ a functor. We can ask when the functor $F$ descends to a functor $F' : D(\mathcal{A}) \longrightarrow D(\mathcal{B})$ when we apply it component-wise to complexes in $D(\mathcal{A})$ (One condition we clearly need, is that $F'$ maps quasi-isomorphic complexes into quasi-isomorphic complexes). This is the case when $F$ is exact, as the following proposition shows:

**Definition 3.1.** *Let $A, B$ be abelian categories and $F : A \longrightarrow B$ an exact functor. Then the functor $F' : D(A) \longrightarrow D(B)$ obtained by applying $F$ component-wise to complexes in $D(A)$ is well-defined and exact in the sense of definition 2.15.*

PROOF. A proof can be found in [**GM03**, III.6.2]. □

While this result gives an answer to our initial question, it is not a very satisfying one: many important functors, like $\mathrm{Hom}(\cdot,\cdot), \cdot\otimes\cdot$ or the global section functor $\Gamma$ for complexes of sheaves are only left or right exact, but not both. In order to address this situation, we introduce left- and right-derived functors. This means that for any left-(right-)exact functor $F : A \longrightarrow B$ such that $A$ contains a class adapted to $F$ we can define an induced functor $RF : D^+(A) \longrightarrow D^+(B)$ ($LF : D^-(A) \longrightarrow D^-(B)$):

**Theorem 3.2.** *Let $F : A \longrightarrow B$ be a left-(right-)exact functor. If $A$ contains a class adapted to $F$, then $F$ induces a unique exact functor $RF : D^+(A) \longrightarrow D^+(B)$ ($LF : D^-(A) \longrightarrow D^-(B)$).*

PROOF. Roughly, the construction works as follows: a class $\mathcal{I}_F$ adapted to $F$ consists of objects that have the following properties:

(1) If $C^\bullet \in \mathcal{K}^+(A)$ ($\in \mathcal{K}^-(A)$ is acyclic with $C^i \in \mathcal{I}_F$, then $F(C^\bullet)$ is acyclic
(2) Any object of $A$ can be embedded into (is the quotient of) an object of $\mathcal{I}_F$.

These properties imply that any $C^\bullet \in \mathcal{K}^+(A)$ ($\in \mathcal{K}^-(A)$ is quasi-isomorphic to some complex $C^\bullet_*$ with $C^i_* \in \mathcal{I}_F$. Now, we can define a functor $RF : RF : D^+(A) \longrightarrow D^+(B)$ ($LF : D^-(A) \longrightarrow D^-(B)$) by putting $RF(C^\bullet) = F(C^\bullet_*)$ where the functor $F$ is applied component-wise. For details on the construction, we refer the reader to [**GM03**, Chapter III.6]. □

**Examples:** Consider the category $\mathrm{Ab}(X)$ of sheaves of abelian groups on a topological space $X$. This category is abelian and it contains enough injectives, which means that it contains a class adapted to *every* left-exact functor with domain $\mathrm{Ab}(X)$. Next, consider the category $D^b(X)$ for $X$ a smooth projective variety. In this case, the class of locally free sheaves on $X$ is adapted to the functor $A \otimes \cdot$ for any coherent sheaf $A$ on $X$ and thus induces an exact endofunctor $A \otimes^L \cdot$.

## 4. Homological algebra and sheaf cohomology

Next, we present some basic results for classical derived functors in the context of sheaf cohomology. In this section, we always refer to derived functors as the ones obtained from taking injective resolutions, applying a left-exact functor and taking the $i$-th cohomology.

**Definition 4.1.** *Let $X, Y$ be objects in an abelian category $\mathcal{A}$. We define*

$$Ext^i_{\mathcal{A}}(X,Y) := Hom_{D^b(\mathcal{A})}(A[0], B[i])$$

*Furthermore, for $E^\bullet, F^\bullet \in D^b(\mathcal{A})$, define*

$$Ext^i(E^\bullet, F^\bullet) := Hom_{D^b(\mathcal{A})}(E^\bullet, F^\bullet[i])$$

**Proposition 4.2.** *$Ext^i_{\mathcal{A}}(X,Y) = 0$ for $i < 0$ and $Ext^0_{\mathcal{A}}(X,Y) = Hom_{\mathcal{A}}(X,Y)$.*

PROOF. cf. [**GM03**, III.5.5]. The latter statement is an easy consequence of the fact that we have an equivalence of categories between the full category of $H^0$-objects of $D(\mathcal{A})$ and $\mathcal{A}$ (cf. Proposition 2.9). □

**Proposition 4.3.** *Let $\mathcal{A}$ be an abelian category with sufficiently many injectives, $X$ an object of $\mathcal{A}$. Then*

$$Ext^i_{\mathcal{A}}(X,\cdot) \cong RHom^i_{\mathcal{A}}(X,\cdot)$$

*where $RHom^i_{\mathcal{A}}(X,\cdot)$ is the classical $i$-th left-derived Hom-functor (i.e. the classical Ext).*

Proof. cf. [**GM03**, III.6.14] □

**Definition 4.4.** *Let $A$ be a sheaf of $\mathcal{O}_X$-modules on a noetherian scheme $X$. Then the functor $\mathcal{E}xt(A,\cdot)$ is defined as the (classical) right-derived functor of $\mathcal{H}om(A,\cdot)$.*

**Remark:** Let $A, B$ be sheaves of $\mathcal{O}_X$-modules on a noetherian scheme $X$. Then $\mathrm{Ext}(A,B)$ is a group, whereas $\mathcal{E}xt(A,B)$ is a sheaf on $X$! Note that in order to be able to define the right derived functors $\mathrm{RHom}(A,\cdot)$ and $\mathrm{R}\mathcal{H}om(A,\cdot)$ we actually need to prove that the category of sheaves of $\mathcal{O}_X$-modules has enough injectives. A proof of this result can be found in [**Har77**, III.2.2].

The global and local Ext-functors are related:

**Proposition 4.5.** *Let $X$ be a noetherian scheme, let $F$ be a coherent sheaf on $X$, let $G$ be any $\mathcal{O}_X$-module and let $x \in X$ be a point. Then we have*

$$\mathcal{E}xt^i(F,G)_x \cong Ext^i_{\mathcal{O}_x}(F_x,G_x)$$

*for all $i \geq 0$, where the right-hand side is Ext over the local ring $\mathcal{O}_x$.*

Proof. cf. [**Har77**, III.6.8] □

The following theorem, known as Serre duality, will be an important tool for our further studies, as it makes possible the definition of the Serre functor in the derived category of coherent sheaves on a smooth projective variety. We state the version that can be found in [**Huy06**, Theorem 3.12].

**Theorem 4.6.** (Serre duality) *Let $X$ be a smooth projective variety of dimension $n$ over a field $k$. For two complexes $\mathcal{E}^\bullet, \mathcal{F}^\bullet \in D^b(X)$ there exists a functorial isomorphism*

$$Ext^i(\mathcal{E}^\bullet,\mathcal{F}^\bullet) \xrightarrow{\sim} Ext^{n-i}(\mathcal{F}^\bullet,\mathcal{E}^\bullet \otimes \omega_X)^*$$

A useful statement that we will need in the context of point-like objects in $D^b(X)$ is the following

**Corollary 4.7.** *Let $x \in X$ and $i_x(A)$ be a skyscraper sheaf in $x$ associated to some $\mathcal{O}_{X,x}$-module $A$. Then*

$$H^i(X,A(x)) = \begin{cases} A & \text{if } i=0 \\ 0 & \text{if } i>0 \end{cases}$$

Proof. As all restriction maps of $i_x(A)$ are either zero or the identity map, the sheaf is flasque. Now the statement follows from [**Har77**, III.2.5]. □

The next statement is a useful property of the tensor product:

**Lemma 4.8.** *Let $X$ be projective variety, $A$ an $\mathcal{O}_X$-module, $B \subset A$ a submodule and $C$ a locally free sheaf on $X$. Then*

$$(A \otimes C)/(B \otimes C) \cong (A/B) \otimes C$$

Proof. We have an exact sequence

$$0 \longrightarrow B \longrightarrow A \longrightarrow A/B \longrightarrow 0$$

and as the functor $D \otimes \cdot$ is acyclic for $D$ a locally free sheaf, the sequence

$$0 \longrightarrow B \otimes C \longrightarrow A \otimes C \longrightarrow (A/B) \otimes C \longrightarrow 0$$

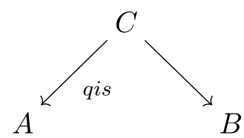
will be exact as well. Now it follows directly that $(A \otimes C)/(B \otimes C) \cong (A/B) \otimes C$. □

## 5. The derived category of an algebraic variety

Here, we state some useful facts about $D^b(X)$ and introduce Serre functors, an important tool for our study of Bondal and Orlov's theorem.

**Theorem 5.1.** *The derived category $D^b(X)$ of coherent sheaves on an smooth projective variety is additive.*

Proof. The difficulty with this theorem is the following: a morphism $A \longrightarrow B$ in $D^b_{\text{coh}}(X)$ can be represented by a diagram in $K(\text{Coh}(X))$ of the following form:




where $A \longrightarrow C$ is a quasi-isomorphism. Now, one needs to find a way to add two such diagrams. One proceeds in the following way: for two diagrams $A \stackrel{\text{qis}}{\longleftarrow} C \stackrel{f}{\longrightarrow} B$ and $A \stackrel{\text{qis}}{\longleftarrow} C' \stackrel{g}{\longrightarrow} B$ it is possible to find a "common denominator" i.e. an object $D$ such that we can find representing diagrams $A \stackrel{\text{qis}}{\longleftarrow} D \stackrel{f'}{\longrightarrow} B$ and $A \stackrel{\text{qis}}{\longleftarrow} D \stackrel{g'}{\longrightarrow} B$. Then define the sum of the diagram as the equivalence class of the diagram $A \stackrel{\text{qis}}{\longleftarrow} D \stackrel{f'+g'}{\longrightarrow} B$. The details of the construction can be found in [**GM03**, III.4.5] □

**Lemma 5.2.** *The derived category $D^b(X)$ of coherent sheaves on an smooth projective variety is $k$-linear and Hom-finite.*

Proof. By the previous theorem, $k$-linearity is an immediate consequence. In order to prove that $D^b(X)$ is Hom-finite, one uses 1.16 and some spectral sequences to prove that $\text{Hom}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is finite-dimensional for all $\mathcal{E}^\bullet, \mathcal{F}^\bullet \in D^b(X)$ (cf. [**Huy06**, Remark 3.7]). □

The following definition is central for the proof of Bondal and Orlov's theorem. In the setting of their theorem, it provides the ability of being able to tensor an element of $D^b(X)$ with an ample invertible sheaf (this will become clear with proposition 5.7).

**Definition 5.3.** *Let $\mathcal{A}$ be a $k$-linear category. A* Serre functor *is a $k$-linear equivalence $S : \mathcal{A} \longrightarrow \mathcal{A}$ such that for any two objects $A, B \in \mathcal{A}$, there exists an isomorphism of $k$-vector spaces*

$$\eta_{A,B} : Hom(A, B) \longrightarrow Hom(B, S(A))^*$$

*which is functorial in $A$ and $B$. (Here, $Hom(B, S(A))^*$ denotes the dual of the $k$-vector space $Hom(B, S(A))$.)*

**Proposition 5.4.** *Let $\mathcal{A}, \mathcal{B}$ be $k$-linear triangulated categories endowed with Serre functors $S_\mathcal{A}$ and $S_\mathcal{B}$ such that all Hom-sets in both categories are finite-dimensional. If $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a $k$-linear equivalence of categories, then there is an isomorphism of functors*

$$F \circ S_\mathcal{A} \cong S_\mathcal{B} \circ F$$

Proof. Let $A, B \in \mathcal{A}$, then we have

$$\mathrm{Hom}(B, A) \cong \mathrm{Hom}(F(B), F(A)) \cong \mathrm{Hom}(F(A), S_\mathcal{B}(F(B)))^*$$

where we first used that $F$ is a $k$-linear equivalence of categories and then applied the Serre functor in $\mathcal{B}$. On the other hand we have $\mathrm{Hom}(B, A) \cong \mathrm{Hom}(A, S_\mathcal{A}(B)))^* \cong \mathrm{Hom}(F(A), F(S_\mathcal{A}(B)))^*$ and as all Hom-spaces are finite-dimensional, we now get a functorial isomorphism $\mathrm{Hom}(F(A), F(S_\mathcal{A}(B)) \cong \mathrm{Hom}(F(A), S_\mathcal{B}(F(B)))$. As $F$ is essentially surjective, for each $B \in \mathcal{A}$ we now have two isomorphic functors $\mathrm{Hom}(\cdot, F(S_\mathcal{A}(B)) \cong \mathrm{Hom}(\cdot, S_\mathcal{B}(F(B))$ from $\mathcal{B}$ to Fun and by the Yoneda-Lemma this means that there is an isomorphism $F(S_\mathcal{A}(B)) \cong S_\mathcal{B}(F(B))$. □

**Corollary 5.5.** *Serre functors are uniquely determined up to isomorphism in $k$-linear, triangulated categories with finite-dimensional Hom-sets.*

Proof. Just plug in $F = \mathrm{id}$ the identity functor in proposition 5.4. □

**Proposition 5.6.** *Any Serre functor on a triangulated category over a field $k$ is exact.*

Proof. cf. [**Huy06**, Proposition 1.46]. In order to prove the first identity for the special case of Hom-finite categories, just use prop. 5.4 with $F = T$, the shift functor. □

**Proposition 5.7.** *For a smooth projective variety $X$ of dimension $n$, the functor $S : D^b(X) \longrightarrow D^b(X)$, $S(C) = C \otimes \omega_X[n]$ is a Serre functor, where $\omega_X$ denotes the canonical bundle of $X$.*

Proof. This is [**Huy06**, Theorem 3.12]. It follows from Serre duality and the fact that $\cdot \otimes \omega_X[n]$ is fully faithful and essentially surjective. Notice the subtlety that a priori, the usual tensor product is not defined on $D^b(X)$ (we would have to resort to the left-derived tensor product $\otimes^L$). However, as $\omega_X$ is locally free, the functor $\omega_X \otimes \cdot$ sends acyclic complexes of coherent sheaves on $X$ to acyclic ones and thus $S$ is well-defined. □

**Remark:** In the setting of the previous proposition, $D^b(X)$ has finite-dimensional Hom-spaces according to lemma 5.2. Thus, corollary 5.5 applies and we see that the functor $S$ from proposition 5.7 is the unique Serre functor on $D^b(X)$.

The next definition is important for the treatment of Bondal and Orlov's theorem as well as for the recovery of Chow groups which we will be looking at later on.

**Definition 5.8.** *For $F \in D^b(X)$, we define the* (cohomological) support *of $F$ by*

$$supp(F) := \bigcup_i supp(\mathcal{H}^i),$$

*where $\mathcal{H}^i$ is the $i$-th cohomology sheaf of $F$.*

**Remark:** By abuse of notation, we write $\mathrm{supp}(F)$ for $F \in D^b(X)$ and $F \in \mathrm{Coh}(X)$. However, there is little trouble involved, as for $G \in \mathrm{Coh}(X)$ we have $\mathrm{supp}(G[0]) = \mathrm{supp}(G)$. Note also that $\mathrm{supp}(F)$ is stable under isomorphism, as isomorphic objects in $D^b(X)$ have isomorphic cohomology sheaves. Finally, this definition is also equivalent to the following one (cf. [**Tho97**, Definition 3.2]):

$$\mathrm{supp}(F) = \{x \in X | F_x \ncong 0 \in D^b(\mathrm{Mod}(\mathcal{O}_{X,x}))\}$$

**Proposition 5.9.** *Let $A^\bullet \in D^b(X)$ be a complex with $m := max\{i|\mathcal{H}^i(A^\bullet) \neq 0\}$ and $n := min\{i|\mathcal{H}^i(A^\bullet) \neq 0\}$. Then there exists an epimorphisms $\varphi : A^\bullet \longrightarrow \mathcal{H}^m(A^\bullet)[-m]$ and a monomorphism $\psi : \mathcal{H}^n(A^\bullet)[-n] \longrightarrow A^\bullet$ such that $\mathcal{H}^m(\varphi) = id = \mathcal{H}^n(\psi)$.*

Proof. From the proof of prop. 2.9 we know that there is a q.i.s. from the complex $C^\bullet$

$$\ldots \longrightarrow A^{m-2} \longrightarrow A^{m-1} \longrightarrow \ker(d^m) \longrightarrow 0 \longrightarrow \ldots$$

to $A^\bullet$. Now we also have an epimorphism of complexes

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & A^{m-2} & \longrightarrow & A^{m-1} & \longrightarrow & \ker(d^m) & \longrightarrow & 0 & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow \pi & & & \\
\cdots \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \ker(d^m)/\mathrm{im}(d^{m-1}) & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}$$

where $\pi$ is the natural projection map $\pi : \ker(d^m) \longrightarrow \ker(d^m)/\mathrm{im}(d^{m-1})$. Thus we have a diagram

$$\varphi : A^\bullet \overset{\text{qis}}{\longleftarrow} C^\bullet \longrightarrow \mathcal{H}^m(A^\bullet)[-m]$$

which proves that there is a morphism $\varphi : A^\bullet \longrightarrow \mathcal{H}^m(A^\bullet)[-m]$ in $D^b(X)$. It is clear from the construction that $\mathcal{H}^m(\varphi) = \mathrm{id}$.

The statement for $\psi : \mathcal{H}^n(A^\bullet)[-n] \longrightarrow A^\bullet$ follows analogously from the fact that there is a q.i.s $A^\bullet \longrightarrow D^\bullet$ for the complex $D^\bullet$

$$\ldots \longrightarrow 0 \longrightarrow \mathrm{coker}(d^{n-1}) \longrightarrow A^{n+1} \longrightarrow A^{n+2} \longrightarrow \ldots$$

and a monomorphism of complexes

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & 0 & \longrightarrow & \mathrm{coker}(d^{n-1}) & \longrightarrow & A^{n+1} & \longrightarrow & A^{n+2} & \longrightarrow \cdots \\
& \uparrow & & \iota \uparrow & & \uparrow & & \uparrow & \\
\cdots \longrightarrow & 0 & \longrightarrow & \ker(d^n)/\mathrm{im}(d^{n-1}) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}$$

where $\iota$ is the inclusion $\ker(d^n)/\mathrm{im}(d^{n-1}) \longrightarrow A^n/\mathrm{im}(d^{n-1})$. Now we got a diagram

$$\psi : A^\bullet \overset{\text{qis}}{\longrightarrow} D^\bullet \longleftarrow \mathcal{H}^n(A^\bullet)[-n]$$

which yields the desired result as in the previous case. □

We will need the following statement:

**Proposition 5.10.** *Let $X$ be a smooth projective variety. Then the skyscraper sheaves $k(x)$, with $x \in X$ a closed point, form a spanning class for $D^b(X)$, i.e.*

(1) *If $Hom(k(x), B[i]) = 0$ for all closed points $x \in X$, $B \in D^b(X)$ and $i \in \mathbb{Z}$ then $B \cong 0$ and*

*(2) if* $Hom(B[i], k(x)) = 0$ *for all closed points* $x \in X$*,* $B \in D^b(X)$ *and* $i \in \mathbb{Z}$ *then* $B \cong 0$*.*

Proof. cf. [**Huy06**, Proposition 3.17] □

**Remark:** As mentioned before, the two conditions are equivalent, as $D^b(X)$ has a Serre functor.

## 6. Spectral sequences

This section is meant as a pure reminder and we just give the very basic definitions we need and the examples that we'll be working with. For our definitions, we follow [**Huy06**, Section 2.3].

**Definition 6.1.** *A* spectral sequence *in an abelian category* $\mathcal{A}$ *is a collection of objects*

$$(E_r^{p,q}, E^n),\ \ n, p, q, r \in \mathbb{Z}, r \geq 1$$

*and morphisms*

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1}$$

*subject to the following conditions:*

*(1)* $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ *which yields a complex* $E_r^{p+\bullet r,q-\bullet r+\bullet}$*.*

*(2) We have isomorphisms* $E_{r+1}^{p,q} \cong H^0(E_r^{p+\bullet r,q-\bullet r+\bullet})$ *which are part of the data.*

*(3) For any* $(p, q)$ *there exists an* $r_0$ *such that* $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$ *for all* $r \geq r_0$*. This implies that* $E_r^{p,q} \cong E_{r_0}^{p,q}$ *for all* $r \geq r_0$*. Denote* $E_{r_0}^{p,q}$ *by* $E_\infty^{p,q}$*.*

*(4) There is a decreasing filtration*

$$F^{p+1}E^n \subset F^pE^n \subset \ldots \subset E^n$$

*such that*

$$\bigcap_p F^pE^n = 0 \text{ and } \bigcup_p F^pE^n = E^n$$

*and there are isomorphisms* $E_\infty^{p,q} \cong F^pE^{p+q}/F^{p+1}E^{p+q}$

If the objects of one layer $E_r^{p,q}$ are explicitly given, one writes

$$E_r^{p,q} \Rightarrow E^{p+q}$$

In many applications, one has an explicit description for the layer $r = 2$.

Spectral sequences naturally arise in the study of the composition of two derived functors. We will not go into details here but rather give two examples that we will use in the proof of Bondal and Orlov's theorem:

**Proposition 6.2.** *Let* $X$ *be a smooth projective variety and* $A, B \in D^b(X)$*. Then there are a spectral sequences*

$$E_2^{p,q} = Hom_{D^b(X)}(H^{-q}(A), B[p]) \Rightarrow Hom_{D^b(X)}(A, B[p+q])$$

*and*

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(A, B)) \Rightarrow Ext^{p+q}(A, B)$$

*where the latter is called the* local-to-global spectral sequence*.*

Proof. Both sequences can be found in [**Huy06**], chapter 2.3 and 3.3. □

**Remark:** A general way to use the sequences is when one knows that $d_r^{p,q} = d_r^{p-r,q+r-1} = 0$ for some $(p,q)$ and all $r$. Then $E_2^{p,q} \neq 0$ implies $E^{p+q} \neq 0$. Another useful fact is that if $E_m^{p,q} = 0$ then $E_n^{p,q} = 0$ for all $n \geq m$, which follows from property 2 of the definition. Spectral sequences are mostly used to obtain information about the objects $E^{p+q}$ as well as the objects $E_\infty^{p,q}$.

To finish this section, we want to give a small but useful application. We prove a result which generalizes 1.5 and uses the local-to-global spectral sequence. It also serves as a rather easy example.

**Lemma 6.3.** *Let $X$ be a smooth projective variety and $x, y \in X$ be two distinct closed points with associated skyscraper sheaves $k(x), k(y)$. Then*

$$Ext^i(k(x), k(y)) = 0 \text{ for all } i \in \mathbb{Z}$$

PROOF. We use the spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(k(x), k(y))) \Rightarrow \mathrm{Ext}^{p+q}(k(x), k(y))$$

First, we will look at the sheaf $\mathcal{E}xt^q(k(x), k(y))$. Define the open subsets $U_x = X \setminus \{x\}$ and $U_y = X \setminus \{y\}$, which cover $X$. Then [**Har77**, III.6.2] tells us that

$$\mathcal{E}xt^q_X(k(x), k(y))|_{U_x} = \mathcal{E}xt^q_{U_x}(k(x)|_{U_x}, k(y)|_{U_x}) = \mathcal{E}xt^q_{U_x}(0, k(y)|_{U_x}) = 0$$

and

$$\mathcal{E}xt^q_X(k(x), k(y))|_{U_y} = \mathcal{E}xt^q_{U_y}(k(x)|_{U_y}, k(y)|_{U_y}) = \mathcal{E}xt^q_{U_y}(k(x)|_{U_y}, 0) = 0$$

which implies $\mathcal{E}xt^q(k(x), k(y)) = 0$ for all $q$ and thus $E_2^{p,q} = 0$ for all $p, q$. This immediately yields $\mathrm{Ext}^i(k(x), k(y)) = 0$ for all $i$ by property (4) of definition 6.1. □

CHAPTER 2

# Bondal and Orlov's theorem

In this section, we will show a proof of Bondal and Orlov's theorem (cf. [**BO01**]) based on D. Huybrechts' version of the proof (cf. [**Huy06**, Chapter 4]). In the following, all schemes considered will be smooth projective varieties over an algebraically closed field $k$. Before we start, here's a short notational note:

**Notation:** If we're dealing with an object $B \in D^b(X)$ and want to stress that it is a complex, we will write $B^\bullet$. If we're talking about the cohomology sheaves of a complex $B^\bullet$ we will write $\mathcal{H}^n(B^\bullet)$, whereas the cohomology of a sheaf $A$ will be denoted by $H^n(A)$.

## 1. A useful proposition

Proposition 1.1 already indicates that some of the geometry of a smooth projective variety is encoded in its derived category of coherent sheaves. Furthermore, it provides information we need to tackle the proof of the main theorem.

**Proposition 1.1.** *Let $X$ and $Y$ be smooth projective varieties over a field $k$. If there exists an exact equivalence*

$$F : D^b(X) \xrightarrow{\sim} D^b(Y)$$

*of their bounded derived categories, then*

$$dim\ X = dim\ Y$$

Proof. Since both varieties are smooth projective, their derived categories of coherent sheaves are endowed with natural Serre functors $S_X$ and $S_Y$, which commute with $F$ by chapter 1, prop. 5.4. Now, pick a closed point $x \in X$. Then $k(x) \cong k(x) \otimes \omega_X = S_X(k(x))[-\dim(X)]$ and thus we can make the calculation

$$\begin{aligned} F(k(x)) &\cong F(S_X(k(x))[-\dim(X)]) \\ &\cong F(S_X(k(x)))[-\dim(X)] \text{ as } F \text{ is exact} \\ &\cong S_Y(F(k(x)))[-\dim(X)] \\ &= F(k(x)) \otimes \omega_Y[\dim(Y) - \dim(X)] \end{aligned}$$

Since $F$ is an equivalence, we know that $F(k(x))$ is a non-trivial and bounded complex in $D^b(Y)$. Hence there we can pick $i$ maximal (resp. minimal) such that $\mathcal{H}^i(F(k(x))) \neq 0$.

Then we make the computation

$$\begin{aligned} 0 &\neq \mathcal{H}^i(F(k(x))) \\ &\cong \mathcal{H}^i(F(k(x)) \otimes \omega_Y[\dim(Y) - \dim(X)]) \\ &= \mathcal{H}^{i+\dim(Y)-\dim(X)}(F(k(x))) \otimes \omega_Y \end{aligned}$$

where the last equality follows from chapter 1, lemma 4.8 as $\omega_Y$ is locally free. This implies that $\mathcal{H}^{i+\dim(Y)-\dim(X)}(F(k(x))) \neq 0$ as well (which we can, for example, check on the stalks of the sheaf). Now we get a contradiction to the maximality (resp. minimality) of $i$ if $\dim(Y) > \dim(X)$ (resp. $\dim(Y) < \dim(X)$) and thus we have $\dim(Y) = \dim(X) := n$.

□

## 2. Point-like and invertible objects

We will introduce some geometric notions to $D^b(X)$ by defining point-like objects and invertible objects. These notions are central for the proof of Bondal and Orlov's theorem. Note that the results we present will not always require the varieties to have ample canonical bundle. We will make use of this fact later on.

**Definition 2.1.** *Let $\mathcal{A}$ be a $k$-linear category with Serre functor $S$. An object $P$ of $\mathcal{A}$ is called* point-like of codimension $d$ *if*

*(i)* $S(P) \cong P[d]$

*(ii)* $Hom(P, P[i]) = 0$ *for* $i < 0$

*(iii)* $k(P) := Hom(P, P)$ *is a field*

*An object that satisfies the last requirement is called* simple.

**Definition 2.2.** *Let $\mathcal{A}$ be a triangulated category with Serre functor $S$. An object $L$ of $\mathcal{A}$ is called* invertible *if for any point-like object $P$ of $\mathcal{A}$, there exists $n_P \in \mathbb{Z}$ such that*

$$Hom(L, P[i]) = \begin{cases} k(P) & \text{if } i = n_P \\ 0 & \text{otherwise} \end{cases}$$

As the naming suggests, point-like objects will correspond to points on the variety $X$ and invertible objects will correspond to invertible sheaves on $X$. Establishing this correspondence is the next part of the proof.

**Proposition 2.3.** *Let $X$ be a smooth projective variety and suppose $\mathcal{F}^\bullet$ is a simple object in $D^b(X)$ with zero-dimensional support. If $Hom(\mathcal{F}^\bullet, \mathcal{F}^\bullet[i]) = 0$ for $i < 0$ then*

$$\mathcal{F}^\bullet \cong k(x)[m]$$

*for some closed point $x \in X$ and some integer $m$.*

Proof. First we show that $\mathcal{F}^\bullet$ is concentrated in one closed point only. We know that $\operatorname{supp}(\mathcal{F}^\bullet) = \bigcup \operatorname{supp}(\mathcal{H}^i(\mathcal{F}^\bullet))$ is a zero-dimensional closed subset of $X$: our variety is noetherian, the cohomology sheaves are coherent and the union is finite as we're working with homologically bounded complexes, and thus definition 1.13 from chapter 1 applies. Then [**Huy06**, Lemma 3.9] tells us that if $\operatorname{supp}(\mathcal{F}^\bullet)$ can be written as $Z_1 \coprod Z_2$ with $Z_1$ and $Z_2$ closed disjoint sets then $\mathcal{F}^\bullet$ can be written as $\mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet$ with

$\operatorname{supp}(\mathcal{F}_1^\bullet) \subset Z_1$ and $\operatorname{supp}(\mathcal{F}_2^\bullet) \subset Z_2$. But now, the projection on either of the two summands is a non-trivial morphism in $D^b(X)$ which is not invertible: indeed, it cannot be a quasi-isomorphism as $\mathcal{H}^n(\mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet) \cong \mathcal{H}^n(\mathcal{F}_1^\bullet) \oplus \mathcal{H}^n(\mathcal{F}_2^\bullet)$ and on the homology-level the projection isn't an isomorphism either unless one of the summands is trivial (which is not the case as both summands have non-trivial support). But this is impossible as $\mathcal{F}^\bullet$ was a simple object and thus we can conclude that all cohomology sheaves of $\mathcal{F}^\bullet$ have support the same closed point $x \in X$.

Now, set $m_0 = \max\{n|\mathcal{H}^n(\mathcal{F}^\bullet) \neq 0\}$ and $m_1 = \min\{n|\mathcal{H}^n(\mathcal{F}^\bullet) \neq 0\}$ (note that this is well-defined as $\mathcal{F}^\bullet$ is bounded) and look at the sheaves $\mathcal{H}^{m_0}(\mathcal{F}^\bullet)$ and $\mathcal{H}^{m_1}(\mathcal{F}^\bullet)$, which are both concentrated in $x$, i.e. $\mathcal{H}^{m_0}(\mathcal{F}^\bullet)|_{X\setminus\{x\}} = \mathcal{H}^{m_1}(\mathcal{F}^\bullet)|_{X\setminus\{x\}} = 0$.

A general fact from commutative algebra now tells us that there is a non-trivial module homomorphism $\mathcal{H}^{m_0}(\mathcal{F}^\bullet)_x \longrightarrow \mathcal{H}^{m_1}(\mathcal{F}^\bullet)_x$: both $\mathcal{H}^{m_0}(\mathcal{F}^\bullet)_x$ and $\mathcal{H}^{m_1}(\mathcal{F}^\bullet)_x$ are finitely generated modules over the local noetherian ring $\mathcal{O}_{X,x}$ with support $\mathfrak{m}_x$. Then there exists a surjection $\mathcal{H}^{m_0}(\mathcal{F}^\bullet)_x \twoheadrightarrow \mathcal{O}_{X,x}/\mathfrak{m}_x = k(x)$ and an injection $\mathcal{O}_{X,x}/\mathfrak{m}_x = k(x) \hookrightarrow \mathcal{H}^{m_1}(\mathcal{F}^\bullet)_x$. Thus we get a non-trivial morphism

$$\mathcal{H}^{m_0}(\mathcal{F}^\bullet) \longrightarrow \mathcal{H}^{m_1}(\mathcal{F}^\bullet)$$

Furthermore there exists an epimorphism $\mathcal{F}^\bullet[m_0] \longrightarrow \mathcal{H}^{m_0}(\mathcal{F}^\bullet)$ and monomorphism $\mathcal{H}^{m_1}(\mathcal{F}^\bullet) \longrightarrow \mathcal{F}^\bullet[m_1]$ by Chapter 1, prop. 5.9 and we can look at the composition

$$\mathcal{F}^\bullet[m_0] \longrightarrow \mathcal{H}^{m_0}(\mathcal{F}^\bullet) \longrightarrow \mathcal{H}^{m_1}(\mathcal{F}^\bullet) \longrightarrow \mathcal{F}^\bullet[m_1]$$

This homomorphism is non-trivial by construction, but by our assumption it must be trivial unless $m_0 \neq m_1$: indeed, we must have $m_0 \geq m_1$ by definition and if we have $m_0 > m_1$, then we will have a morphism $\mathcal{F}^\bullet \longrightarrow \mathcal{F}^\bullet[m_1 - m_0]$ which must be the trivial homomorphism by assumption. Thus we have that $m_1 = m_2 =: m$ and we see that $\mathcal{F}^\bullet$ has only one non-trivial cohomology sheaf. This implies that $\mathcal{F}^\bullet \cong \mathcal{F}[m]$ in $D^b(X)$, where $\mathcal{F}$ is a skyscraper sheaf in $x$. The only such sheaf which is also simple is $k(x)$: Indeed, we have a surjection $\mathcal{F} \longrightarrow k(x)$ and an injection $k(x) \longrightarrow \mathcal{F}$. If $\mathcal{F}$ is not equal to $k(x)$, the composition of these maps will be a non-invertible element of $\operatorname{Hom}(\mathcal{F}, \mathcal{F})$ which contradicts its simplicity. □

**Proposition 2.4.** *Let $X$ be a smooth projective variety of dimension $n$ and suppose that $\omega_X$ is ample. Then the point-like objects in $D^b(X)$ are exactly the objects which are isomorphic to $k(x)[m]$, where $x \in X$ is a closed point and $m \in \mathbb{Z}$.*

Proof. We can check that any object of the form $k(x)[m]$ is a point-like object, even if $\omega_X$ is not ample: first we need to check that $k(x)[m] \otimes \omega_X[n] \cong k(x)[m+n]$. We know that $\omega_X$ is a locally free $\mathcal{O}_X$-module of rank 1 and as we're tensoring over $\mathcal{O}_X$, it is clear that $k(x) \otimes \omega_X \cong k(x)$. Thus $k(x)[m] \otimes \omega_X[n] \cong k(x)[m][n] = k(x)[m+n]$. Furthermore we clearly have $\operatorname{Hom}(k(x), k(x)[n]) = \operatorname{Ext}^n(k(x), k(x)) = 0$ for all $n \in \mathbb{Z}_{<0}$ as negative Ext-groups are zero for every coherent sheaf (for a much more general statement cf. [**GM03**, III.5.5]). It remains to prove that $\operatorname{Hom}_{D^b(X)}(k(x), k(x)) = K$, where $K$ is a field. Finding an element in $\operatorname{Hom}_{D^b(X)}(k(x), k(x)) = \operatorname{Ext}^0(k(x), k(x)) = \operatorname{Hom}(k(x), k(x))$ is equivalent to finding a $k(x)$-linear map $k(x) \longrightarrow k(x)$. It is clear that all of these maps are precisely of the form "multiplication by an element of $k(x)$" which proves that $\operatorname{Hom}_{D^b(X)}(k(x)[m], k(x)[m]) = k(x)$.

Now assume that $P \in D^b(X)$ is a point-like object of codimension $d$, with cohomology sheaves $\mathcal{H}^i$ which are not all zero (we can assume this w.l.o.g. as otherwise, $P$ would be isomorphic to $0 \in D^b(X)$, which is certainly not a point-like object). The Serre functor in $D^b(X)$ is given by $\mathcal{F}^\bullet \mapsto \mathcal{F}^\bullet \otimes \omega_X[n]$ and as $P$ is a point-like object we have that $P \otimes \omega_X[n] \cong P[d]$. This implies that $\mathcal{H}^{i-n} \otimes \omega_X \cong \mathcal{H}^{i-d} \Leftrightarrow \mathcal{H}^{i+d-n} \otimes \omega_X \cong \mathcal{H}^i$ for all $i$ as taking cohomology commutes with tensoring by a locally free sheaf by chapter 1, lemma 4.8. As our complexes are homologically bounded, we must have $b_1 < b_2 \in \mathbb{Z}$ such that $b_1 = \max\{z \in \mathbb{Z} | \mathcal{H}^a = 0 \ \forall a < z\}$ and $b_2 = \min\{z \in \mathbb{Z} | \mathcal{H}^a = 0 \ \forall a > z\}$ (as $P$ has non-trivial cohomology sheaves, this means that $\mathcal{H}^{b_1} \neq 0 \neq \mathcal{H}^{b_2}$). Now if $(d-n) > 0$, we have that $\mathcal{H}^{b_2} \cong \mathcal{H}^{b_2+d-n} \otimes \omega_X = 0 \otimes \omega_X = 0$, which is a contradiction. By an analogous argument, one shows that we cannot have $(d-n) < 0$ either and thus $d = n$ and therefore $\mathcal{H}^i \otimes \omega_X \cong \mathcal{H}^i$ for all $i$.

We can now prove that $\mathcal{H}^i$ has 0-dimensional support. As $\omega_X$ is ample, we know there is a $k \in \mathbb{Z}_{>0}$ such that $\omega_X^{\otimes k}$ is very ample, i.e. there is a closed immersion $i : X \longrightarrow \mathbb{P}^n_k$ such that $\omega_X^{\otimes k} = i^*(\mathcal{O}(1))$. This means that the Hilbert polynomial of $\mathcal{H}^i$ relative to the embedding induced by $\omega_X^{\otimes k}$ is given by $P_{\mathcal{H}^i}(n) = \chi(\mathcal{H}^i \otimes \omega_X^{\otimes k \cdot n}) = \chi(\mathcal{H}^i)$ by what we've shown earlier. But this means that $P_{\mathcal{H}^i}$ has degree 0 which implies, according to [Debarre: Higher-dimensional algebraic geometry, Theorem 1.5], that $\operatorname{supp}(\mathcal{H}^i)$ can have at most dimension 0.

As $\operatorname{supp}(P) = \bigcup_i \operatorname{supp}(\mathcal{H}^i)$ and this union has only finitely many non-empty terms, we see that $P$ has support in dimension zero as well and thus we can apply the previous proposition to obtain the result. □

Next, we show that invertible objects correspond to line bundles:

**Proposition 2.5.** *Let $X$ be a smooth projective variety. Any invertible object of $D^b(X)$ is of the form $\mathcal{L}[m]$ with $\mathcal{L}$ a line bundle on $X$ and $m \in \mathbb{Z}$. Conversely, if $\omega_X$ or $\omega_X^*$ is ample, then for any line bundle $\mathcal{L}$ and any $m \in \mathbb{Z}$, the object $\mathcal{L}[m]$ is invertible.*

Proof. Suppose that $L$ is an invertible object in $D^b(X)$ and let $m$ be maximal with $\mathcal{H}^m := \mathcal{H}^m(L) \not\cong 0$.

*Step 1: $Ext^1(\mathcal{H}^m, k(x_0)) = 0$ for all $x_0 \in supp(\mathcal{H}^m)$.* By chapter 1, proposition 5.9 there exists a natural morphism $\phi : L \longrightarrow \mathcal{H}^m[-m]$ which induces the identity on the $m$-th cohomology. Next, pick a point $x_0 \in \operatorname{supp}(\mathcal{H}^m)$. Then there exists a non-trivial homomorphism $\mathcal{H}^m \longrightarrow k(x_0)$ (take for example the composition of the map $\mathcal{H}^m \longrightarrow \mathcal{H}^m|_{x_0}$ followed by a quotient map $\mathcal{H}^m|_{x_0} \longrightarrow k(x_0)$) and thus we have that $\operatorname{Hom}(\mathcal{H}^m, k(x_0)) \neq 0$. On the the other hand, notice that $\operatorname{Hom}(\mathcal{H}^m, k(x_0)) = \operatorname{Hom}(L, k(x_0)[-m])$: we know that a morphism between $L$ and $k(x_0)[-m]$ in $D^b(X)$ is given by a diagram $L \overset{qis}{\longleftarrow} K \longrightarrow k(x_0)[-m]$. Let $K$ be such a complex, then a morphism of complexes $\alpha : K \longrightarrow k(x_0)[-m]$ looks like this:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & K^{m-1} & \xrightarrow{d^{m-1}} & K^m & \xrightarrow{d^m} & K^{m+1} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle \alpha^m} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & k(x_0) & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

This means that $\alpha$ is completely determined by $\alpha^m$. But from the diagram we see that $\text{im}(d^{m-1}) \subset \ker\alpha^m$ and thus $\alpha_m$ is well-defined on $\mathcal{H}^m(K) \cong \mathcal{H}^m$ as $K$ and $L$ are quasi-isomorphic. This gives a map $\text{Hom}(L, k(x_0)[-m]) \longrightarrow \text{Hom}(\mathcal{H}^m, k(x_0))$. On the other hand, let $\beta \in \text{Hom}(\mathcal{H}^m, k(x_0))$, then $\beta$ produces a map of complexes $\beta' : L' \longrightarrow k(x_0)[-m]$ given by the diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & L^{m-1} & \xrightarrow{d^{m-1}} & \ker d^m & \xrightarrow{d^m} & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle \beta''} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & k(x_0) & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

where $\beta''$ is the composition of the canonical projection $\ker d^m \longrightarrow \mathcal{H}^m$ and $\beta$, and $L'$ is a complex that is quasi-isomorphic to $L$. Thus we have a diagram $L \xleftarrow{qis} L' \longrightarrow k(x_0)[-m]$ which is a morphism $\beta' : L \longrightarrow k(x_0)[-m]$ in $D^b(X)$. This defines a map $\text{Hom}(\mathcal{H}^m, k(x_0)) \longrightarrow \text{Hom}(L, k(x_0)[-m])$.

Clearly, the two processes described above are inverse to each other, and we get that $\text{Hom}(\mathcal{H}^m, k(x_0)) = \text{Hom}(L, k(x_0)[-m])$. Therefore we also have $\text{Hom}(L, k(x_0)[-m]) \neq 0$ and as $L$ was an invertible object we obtain that $n_{k(x_0)} = -m$. Now we can conclude that $\text{Hom}(\mathcal{H}^m, k(x_0)[1]) = \text{Hom}(L, k(x_0)[1+n_{k(x_0)}]) = 0$ and we have that $\text{Ext}^1(\mathcal{H}^m, k(x_0)) = \text{Hom}(\mathcal{H}^m, k(x_0)[1]) = 0$ (according to definition 4.1 from chapter 1) as soon as $x_0$ is in the support of $\mathcal{H}^m$.

*Step 2: $\mathcal{H}^m$ is an invertible sheaf on $X$.* We use the local-to-global spectral sequence from chapter 1, 6.2:

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{H}^m, k(x_0))) \Rightarrow \text{Ext}^{p+q}(\mathcal{H}^m, k(x_0))$$

Recall the definition of $\mathcal{E}xt^q(\mathcal{H}^m, k(x_0))$ from chapter 1. First we want to prove that $E_2^{2,0} = H^2(X, \mathcal{E}xt^0(\mathcal{H}^m, k(x_0)) = 0$. Notice that $\mathcal{E}xt^0(\mathcal{H}^m, k(x_0)) \cong \mathcal{H}om(\mathcal{H}^m, k(x_0))$ and thus the the sheaf $\mathcal{E}xt^0(\mathcal{H}^m, k(x_0))$ is concentrated in $x_0$: indeed, let $U \subset X$ be an open subset of $X$, then

$$\mathcal{E}xt^0(\mathcal{H}^m, k(x_0))(U) = \text{Hom}(\mathcal{H}^m|_U, k(x_0)|_U) = 0 \text{ if } x_0 \notin U$$

and thus we must have $\text{supp}(\mathcal{E}xt^0(\mathcal{H}^m, k(x_0))) = \{x_0\}$. It now follows from chapter 1, lemma 1.14 that $\mathcal{E}xt^0(\mathcal{H}^m, k(x_0))$ is a subsheaf of the skyscraper sheaf associated to the $\mathcal{O}_{X,x_0}$-module $\mathcal{H}om(\mathcal{H}^m, k(x_0))_{x_0}$ and it particularly is a skyscraper sheaf itself. Using chapter 1, prop. 4.7 we find that $H^2(X, \mathcal{E}xt^0(\mathcal{H}^m, k(x_0)) = 0$. This proves that

$$E_2^{2,0} = H^2(X, \mathcal{E}xt^0(\mathcal{H}^m, k(x_0)) = 0$$

Now we can conclude that $E_2^{0,1} = E_\infty^{0,1}$: we have $E_2^{-2,2} = H^{-2}(X, \mathcal{E}xt^2(\mathcal{H}^m, k(x_0))) = 0$ by definition and we just proved that $E_2^{2,0} = 0$. For higher sheets of the spectral sequence, the differentials must be zero as negative $\mathcal{E}xt$s and negative cohomology are zero by definition, which indeed implies $E_2^{0,1} = E_\infty^{0,1}$ (here we use that if we have $E_2^{p,q} = 0$, then we must have $E_r^{p,q} = 0$ for all $r \geq 2$). Thus, we have that $E_2^{0,1}$ is a subquotient of $\text{Ext}^1(\mathcal{H}^m, k(x_0))$. But as we proved earlier, $\text{Ext}^1(\mathcal{H}^m, k(x_0)) = 0$ which implies that $E_2^{0,1} = H^0(X, \mathcal{E}xt^1(\mathcal{H}^m, k(x_0))) \cong \Gamma\left(\mathcal{E}xt^1(\mathcal{H}^m, k(x_0))\right) = 0$. $\mathcal{E}xt^1(\mathcal{H}^m, k(x_0)))$ is concentrated in $x_0$, as by chapter 1, proposition 4.5 we have

$$\mathcal{E}xt^1(\mathcal{H}^m, k(x_0)))_y = \mathrm{Ext}^1_{\mathcal{O}_{X,y}}(\mathcal{H}^m_y, k(x_0)_y) = \mathrm{Ext}^1_{\mathcal{O}_{X,y}}(\mathcal{H}^m_y, 0) = 0$$

if $y \neq x_0$ and thus it is a globally generated sheaf. Therefore we also have that $0 = \mathcal{E}xt^1(\mathcal{H}^m, k(x_0)))_{x_0} \cong \mathrm{Ext}^1_{\mathcal{O}_{X,x_0}}(\mathcal{H}^m_{x_0}, k(x_0))$. By a well-known result in commutative algebra (cf. [N. Bourbaki: Algèbre commutative. Chapitre 10, X.3 Prop.4]), this means that $\mathcal{H}^m_{x_0}$ is free over $\mathcal{O}_{X,x_0}$.

Next, recall that $X$ is irreducible (by definition). Therefore we must have that $\mathrm{supp}(\mathcal{H}^m) = X$: indeed, we know that if $\mathcal{H}^m_p$ is free over $\mathcal{O}_{X,p}$ for some $p \in X$, we know that there is some open neighbourhood $U$ of $p$, such that $\mathcal{H}^m|_U$ is free (cf. [**Har77**, Exercise II.5.7]), i.e. $\mathcal{H}^m|_U \cong \oplus_i \mathcal{O}_X|_U$. But then for all points $q \in U$, we have that $\mathcal{H}^m_q = \oplus_i \mathcal{O}_{X,q} \neq 0$. This implies that $\mathrm{supp}(\mathcal{H}^m) = X$ as it is an open and closed subset of an irreducible variety. At this point, we have proved that $\mathcal{H}^m$ is a locally free sheaf on $X$.

Thus, we know that for any $x \in X$ we have $0 \neq \mathrm{Hom}(\mathcal{H}^m, k(x)) = \mathrm{Hom}(L, k(x)[-m])$. This means in particular that $n_{k(x)}$ does not depend on $x$. As $L$ is invertible and $k(x)[m]$ point-like, we have for any point $x \in X$ that $k(x) = \mathrm{Hom}(\mathcal{H}^m, k(x)) = \mathrm{Hom}(L, k(x)[-m])$. This implies that $\mathcal{H}^m_x$ must have rank 1 for any $x \in X$, as otherwise there would be non-invertible homomorphisms. This proves that $\mathcal{H}^m$ is a line bundle.

*Step 3: $\mathcal{H}^i = 0$ for all $i \neq m$.* Our strategy to finish the proof is to show that we actually have $L \cong \mathcal{H}^m[-m]$ in $D^b(X)$. This amounts to proving that $\mathcal{H}^i = 0$ for $i < m$ (as we already have $\mathcal{H}^i = 0$ for $i > m$ by assumption). For this, we take a look at the first spectral sequence from chapter 1, proposition 6.2:

$$E_2^{p,q} = \mathrm{Hom}(\mathcal{H}^{-q}(L), k(x)[p]) \Rightarrow \mathrm{Hom}(L, k(x)[p+q])$$

Notice that $\mathrm{Hom}_{D^b(X)}(\mathcal{H}^{-q}, k(x)[p]) \cong \mathrm{Ext}^p(\mathcal{H}^{-q}, k(x))$ by definition, in particular we have

$$E_2^{p,-m} = \mathrm{Hom}_{D^b(X)}(\mathcal{H}^m, k(x)[p]) \cong \mathrm{Ext}^p(\mathcal{H}^m, k(x))$$

Observe that

$$\begin{aligned} \mathrm{Ext}^p(\mathcal{H}^m, k(x)) &= \mathrm{Ext}^p(\mathcal{O}_X \otimes \mathcal{H}^m, k(x)) \\ &= \mathrm{Ext}^p(\mathcal{O}_X, \mathcal{H}^{m*} \otimes k(x)) \text{ as } \mathcal{H}^m \text{ is locally free, cf. [\textbf{Har77}, III.6.7]} \\ &= H^p(X, \mathcal{H}^{m*} \otimes k(x)) \text{ cf. [\textbf{Har77}, III.6.3]} \end{aligned}$$

where $\mathcal{H}^{m*}$ is the dual sheaf $\mathcal{H}om(\mathcal{H}^m, \mathcal{O}_X)$. Now, we clearly have that $\mathrm{supp}(\mathcal{H}^{m*} \otimes k(x)) = x$ and thus, it is a skyscraper sheaf, according to chapter 1, 1.14. But now proposition 4.7 from chapter 1 says that $H^p(X, \mathcal{H}^{m*} \otimes k(x)) = 0$ for $p \neq 0$. Therefore we know that the complete row $E_2^{p,-m}, p \in \mathbb{Z}$ of our spectral sequence is trivial except for $p = 0$. We now finish the proof by induction: First we show that $\mathcal{H}^{m-1} = 0$ as an induction base. Thus, look at $E_2^{0,-m+1} = \mathrm{Hom}_{D^b(X)}(\mathcal{H}^{m-1}, k(x)[0]) = \mathrm{Ext}^0(\mathcal{H}^{m-1}, k(x))$. We want to prove that $E_2^{0,-m+1} = E_\infty^{0,-m+1}$. In order to do so, notice that $E_2^{r,-m+1-r+1} = 0$ for $r \geq 2$. Indeed, for $r = 2$, our previous result guarantees the claim, and we know that $\mathcal{H}^i = 0$ for $i > m$ and thus we have the result for $r \geq 2$. This implies that on all sheets of the spectral sequence we have that $d : E_r^{0,-m+1} \longrightarrow E_r^{r,-m+1-r+1}$ is the zero map (again, we use that $E_2^{p,q} = 0$ implies that $E_r^{p,q} = 0$ for all $r \geq 2$.). Furthermore, we have that $E_2^{-r,-m+1+r-1} = \mathrm{Ext}^{-r}(\mathcal{H}^{m-r}, k(x)) = 0$ as negative Ext groups are zero by definition.

This means that on all sheets of the spectral sequence one has $d : E_r^{-r,-m+1+r-1} \longrightarrow E_r^{0,-m+1}$ is the zero map and therefore we must have $E_2^{0,-m+1} = E_\infty^{0,-m+1}$. Recall that the spectral sequence converges to $\mathrm{Hom}(L, k(x)[p+q])$ and thus we have that $E_\infty^{0,-m+1}$ is a subquotient of $\mathrm{Hom}(L, k(x)[-m+1])$. But $\mathrm{Hom}(L, k(x)[-m+1]) = 0$ as $L$ is invertible and $k(x)$ is a point-like object with $n_{k(x)} = -m$. This implies $E_\infty^{0,-m+1} = \mathrm{Ext}^0(\mathcal{H}^{m-1}, k(x)) = \mathrm{Hom}(\mathcal{H}^{m-1}, k(x)) = 0$ for all $x \in X$ closed. As all objects of the form $k(x)$ form a spanning class (cf. Proposition 5.10 from chapter 1), we must have $\mathcal{H}^{m-1} = 0$.

Now suppose we have proven $\mathcal{H}^i = 0$ for $i \in \{i_0+1, \ldots, m-1\}$. Then we have that $E_2^{0,-i_0} = E_\infty^{0,-i_0}$: again, we prove that the differentials $E_r^{0,-i_0} \longrightarrow E_r^{r,-i_0-r+1}$ and $E_r^{-r,-i_0+r-1} \longrightarrow E_r^{0,-i_0}$ are trivial. In the first case, this follows from the induction hypothesis, the claim that $E_2^{p,-m} = 0, p \in \mathbb{Z} \setminus \{0\}$ and the fact that $\mathcal{H}^i = 0$ for $i > m$. The second case again follows from the fact that negative Ext groups are zero. Now, by the very same argument as before, we can conclude that $\mathrm{Hom}(\mathcal{H}^{i_0}, k(x)) = 0$ for all $x \in X$ closed, which implies $\mathcal{H}^{i_0} = 0$.

Thus, we proved that $L \cong \mathcal{H}^m[m]$ in $D^b(X)$ which implies our first assertion that any invertible object in $D^b(X)$ is of the form $\mathcal{L}[m]$, where $\mathcal{L}$ is a line bundle.

*Step 4: invertible sheaves correspond to invertible objects.* For the converse, we make the additional assumption that $X$ has ample (anti)-canonical bundle. This enables us to use proposition 2.4, i.e. we can assume that point-like objects in $D^b(X)$ are of the form $k(x)[s]$ for some closed point $x \in X$ and some $s \in \mathbb{Z}$. Therefore, for a line bundle $\mathcal{L}$, we have

$$\begin{aligned}
\mathrm{Hom}(\mathcal{L}[m], P[i]) &= \mathrm{Hom}(\mathcal{L}[m], k(x)[i+s]) \\
&= \mathrm{Ext}^{i+s-m}(\mathcal{L}, k(x)) \\
&\cong \mathrm{Ext}^{i+s-m}(\mathcal{O}_X, \mathcal{L}^* \otimes k(x)) \text{ as } \mathcal{H}^m \text{ is loc. free, cf. [\textbf{Har77}, III.6.7]} \\
&\cong H^{i+s-m}(X, \mathcal{L}^* \otimes k(x)) \text{ cf. [\textbf{Har77}, III.6.3]}
\end{aligned}$$

Now, if $i + s - m \neq 0$, then this expression is 0 by chapter 1, 4.7 as $\mathcal{L}^* \otimes k(x)$ is a skyscraper sheaf. If $m = i + s$, then $\mathrm{Hom}(\mathcal{L}[m], k(x)[i+s]) = \mathrm{Hom}(\mathcal{L}, k(x)) = k$ as $\mathcal{L}$ is locally free of rank 1: indeed, we can cover $X$ with opens $U_i$ such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_X|_{U_i}$. Now pick one $j$ such that $x \in U_j$, then a homomorphism $\mathcal{O}_X|_{U_j} \longrightarrow k(x)$ is completely determined by the image of $1 \in \mathcal{O}_X|_{U_j}$. All other $U_i$ that contain $x$ have non-empty intersection with $U_j$ and as the restriction homomorphism $U_i \longrightarrow U_i \cap U_j$ sends 1 to 1, the homomorphism extends to $U_i$ by sending $1 \in U_j$ to the same element of $k$ as $1 \in U_i$. On the $U_i$ that don't contain $x$, the homomorphism is just 0. Thus, we see that $\mathrm{Hom}(\mathcal{L}, k(x)) = k$.

Thus, if we set $n_P = m - s$ we have proved the claim. This finishes the proof of the theorem. □

## 3. Bondal and Orlov's theorem

Now, we're in a position to tackle the theorem of Bondal and Orlov. We will make use of the fact that we have found intrinsic definitions of objects corresponding to points and line bundles on $X$. An exact equivalence of categories $D^b(X) \longrightarrow D^b(Y)$ will enable us to carry these geometric structures of $X$ over to $Y$. As we will see, the main difficulty of the remaining part is to deduce ampleness of the canonical bundle of $Y$ from the ampleness of the canonical bundle of $X$.

**Theorem 3.1.** *Let $X$ and $Y$ be smooth projective varieties and assume that the (anti)-canonical bundle of $X$ is ample. If there exists an exact equivalence $D^b(X) \cong D^b(Y)$, then $X$ and $Y$ are isomorphic.*

Proof. First, we will give the main idea of the proof: From proposition 1.1 we know that $\dim X = \dim Y =: n$. Furthermore, make the assumption that $F$ maps $\mathcal{O}_X$ to $\mathcal{O}_Y$. Thus, for any $k \in \mathbb{N}$, we can make the calculation

$$F(\omega_X^{\otimes k}) = F(\mathcal{O}_X \otimes \omega_X^{\otimes k}) = F(S_X^k(\mathcal{O}_X))[-kn] = S_Y^k(F(\mathcal{O}_X))[-kn] = S_Y^k(\mathcal{O}_Y)[-kn] = \omega_Y^{\otimes k}$$

where we used that the shift functor and the Serre functor commute with an equivalence of categories. For the next step, notice that

$$\mathrm{Hom}(\mathcal{O}_X, \omega_X^k) = \Gamma(X, \omega_X^k) \text{ (cf. Chapter 1, Lemma 1.6)}$$

Now we can make the computation

$$\begin{aligned} H^0(X, \omega_X^{\otimes k}) &= \Gamma(X, \omega_X^{\otimes k}) \\ &\cong \mathrm{Hom}(\mathcal{O}_X, \omega_X^{\otimes k}) \\ &\cong \mathrm{Hom}(F(\mathcal{O}_X), F(\omega_X^{\otimes k})) \\ &= \mathrm{Hom}(\mathcal{O}_Y, \omega_Y^{\otimes k}) \\ &\cong \Gamma(Y, \omega_Y^{\otimes k}) \\ &= H^0(Y, \omega_Y^{\otimes k}) \end{aligned}$$

Notice that multiplication in the canonical ring $\bigoplus_k H^0(X, \omega_X^{\otimes k})$ is induced by the Serre functor: indeed, as $S_X^k$ is an equivalence of categories, we have that

$$\begin{aligned} \Gamma(X, \omega_X^m) &= \mathrm{Hom}(\mathcal{O}_X, \omega_X^m) \\ &= \mathrm{Hom}(\mathcal{O}_X[0], \omega_X^m[0]) \\ &\cong \mathrm{Hom}(S^k(\mathcal{O}_X[0])[-kn], S^k(\omega_X^m[0])[-kn]) \\ &= \mathrm{Hom}(\omega_X^k[0], \omega_X^{m+k}[0]) \\ &= \mathrm{Hom}(\omega_X^k, \omega_X^{m+k}) \end{aligned}$$

Thus, for $s_1 \in \Gamma(X, \omega_X^{k_1}), s_2 \in \Gamma(X, \omega_X^{k_2})$, we can consider $s_1 \in \mathrm{Hom}(\mathcal{O}_X, \omega_X^{k_1})$ and $s_2 \in \mathrm{Hom}(\omega_X^{k_1}, \omega_X^{k_1+k_2})$ and define $s_1 \cdot s_2 = s_2 \circ s_1 \in \mathrm{Hom}(\mathcal{O}_X, \omega_X^{k_1+k_2}) = \Gamma(X, \omega_X^{k_1+k_2})$. This multiplication coincides with the usual one on the canonical ring, as it is just concatenation of tensors. Therefore $F$ induces a ring-isomorphism and we have that

$$\bigoplus_k H^0(X, \omega_X^{\otimes k}) \cong \bigoplus_k H^0(Y, \omega_Y^{\otimes k})$$

If the (anti-)canonical bundle of $Y$ is also ample and there is an integer $d$ such that $\omega_X^{\otimes d}$ and $\omega_Y^{\otimes d}$ are very ample, this gives

$$X = \text{Proj}\left(\bigoplus_k H^0(X, \omega_X^{\otimes dk})\right) \cong \text{Proj}\left(\bigoplus_k H^0(Y, \omega_Y^{\otimes dk})\right) = Y$$

by theorem 1.15 from chapter 1. So, in order to prove the theorem, we have to show:

(i) $F(\mathcal{O}_X) = \mathcal{O}_Y$
(ii) The (anti-)canonical bundle of $Y$ is ample and of the same order as the canonical bundle of $X$.

We start our proof of $(i)$ by stating that the equivalence $F$ induces bijections between the point like objects of $X$ and $Y$ and the invertible objects of $X$ and $Y$. Indeed, let $P \in D^b(X)$ be a point-like object, then $F(P)$ also is: We have $S_Y(F(P)) = F(S_X(P)) = F(P[d]) = F(P)[d]$, furthermore $\text{Hom}(F(P), F(P[i])) \cong \text{Hom}(P, P[i]) = 0$ for $i < 0$ and equal to $k(P)$ if $i = 0$. As $F$ is an equivalence, this induces a bijection on the isomorphism classes of point-like objects in $D^b(X)$ and $D^b(Y)$. For $L \in D^b(X)$ an invertible object, and $P \in D^b(X)$ a point-like object we have that

$$\text{Hom}(F(L), F(P[i])) \cong \text{Hom}(L, P[i]) = \begin{cases} k(P) & \text{for } i = n_P \\ 0 & \text{otherwise} \end{cases}$$

which proves that $F(L)$ is an invertible object in $D^b(Y)$. As $F$ is an equivalence of categories, we again obtain a bijection on the isomorphism classes of invertible objects.

By proposition 2.5, we have that all objects of the form $L[m]$ with $L$ a line bundle on $X$ are invertible objects on $X$, as $X$ has ample (anti-)canonical bundle. In particular, $\mathcal{O}_X$ defines an invertible objects and thus $F(\mathcal{O}_X)$ is a line bundle on $Y$, which is of the form $M[m]$ for some line bundle $M$ on $Y$, again due to proposition 2.5. Next we compose $F$ with the two equivalences $M^* \otimes \cdot$ (notice that we *don't* need the left-derived tensor product here as $M^*$ is locally free!) and shifting by $-m$ to obtain an equivalence $F'$ which obviously satisfies $F(\mathcal{O}_X) = \mathcal{O}_Y$. This new functor is still an exact equivalence of categories and has property (i), which means that we can assume it without loss of generality. Now, the proof is finished by the following lemma. □

**Lemma 3.2.** *Let $X$ and $Y$ be smooth projective varieties and assume that the (anti)-canonical bundle of $X$ is ample, i.e. there is an integer $d$ such that $\omega_X^{\otimes d}$ is very ample. If there exists an exact equivalence $D^b(X) \cong D^b(Y)$ then $\omega_Y^{\otimes d}$ is very ample.*

Proof. In order to prove that $\omega_Y^d$ is very ample, we first prove that all point-like objects in $D^b(Y)$ are of the form $k(y)[m]$, for $y \in Y$ a closed point. In general, any object of the form $k(y)[m] \in D^b(Y)$ is point-like (prop. 2.4) and as $F$ induces bijections between the sets of point like objects on $D^b(X)$ and $D^b(Y)$ we can find for each object $k(y)[m] \in D^b(Y)$ an closed point $x \in X$ such that $F(k(x)[n]) = k(y)[m]$ for some $n \in \mathbb{Z}$. Now suppose there is a point-like object $P \in D^b(Y)$, which is not of the form $k(y)[m]$ for some closed point $y \in Y$ and denote by $x_P$ the closed point of $X$ such that

$F(k(x_P)[m_P]) = P$ for $m_p \in \mathbb{Z}$. For all $y \in Y$ and $m \in \mathbb{Z}$ we have

$$\begin{aligned}\mathrm{Hom}(P, k(y)[m]) &= \mathrm{Hom}(F(k(x_P)[m_P]), F(k(x_y)[m_y + m]))\\ &= \mathrm{Hom}(k(x_P)[m_P], k(x_y)[m_y + m])\\ &= \mathrm{Hom}(k(x_P), k(x_y)[m_y + m - m_P])\\ &= \mathrm{Ext}^{m_y+m-m_P}(k(x_P), k(x_y)) = 0 \text{ by chapter 1, lemma 6.3}\end{aligned}$$

As the objects of the form $k(y)[m]$ form a spanning class in $D^b(Y)$, we must have $P \cong 0$, which is in contradiction to the fact that it is a point-like object. Thus all point-like objects of $D^b(Y)$ have the form $k(y)[m]$ for some closed point $y \in Y$.

Note that this also implies that for any closed point $x \in X$ there exists a closed point $y \in Y$ such that $F(k(x)) \cong k(y)$ (no shifts needed!): to show this, note that

$$0 \neq \mathrm{Hom}(\mathcal{O}_X, k(x)) = \mathrm{Hom}(F(\mathcal{O}_X), F(k(x))) = \mathrm{Hom}(\mathcal{O}_Y, k(y)[m]) = \mathrm{Ext}^m(\mathcal{O}_Y, k(y))$$

This immediately implies that $m \geq 0$ and as we have that $\mathrm{Ext}^m(\mathcal{O}_Y, k(y)) = H^m(X, k(y))$ and $k(y)$ is flasque, we get that $m = 0$ by chapter 1, 4.7. Thus $F(k(x)) = k(y)$.

Next, we show that $\omega_Y^{\otimes d}$ is very ample. We will do this by using lemma 1.19 from chapter 1 and first show that $\omega_Y^{\otimes d}$ separates points. Thus, let $\phi : \omega_Y^{\otimes k} \longrightarrow k(P) \oplus k(Q)$ be induced by the restrictions $\omega_Y^{\otimes k} \longrightarrow (\omega_Y^{\otimes k})_P \cong \mathcal{O}_{Y,P}$ and $\omega_Y^{\otimes k} \longrightarrow (\omega_Y^{\otimes k})_Q \cong \mathcal{O}_{Y,Q}$ and the projections $\mathcal{O}_{Y,P} \longrightarrow k(P)$ and $\mathcal{O}_{Y,Q} \longrightarrow k(Q)$.

We have that

$$\begin{aligned}\phi \in \mathrm{Hom}(\omega_Y^{\otimes k}, k(P) \oplus k(Q)) &= \mathrm{Hom}(F(\omega_X^{\otimes k}), F(k(P')) \oplus F(k(Q')))\\ &\cong \mathrm{Hom}(\omega_X^{\otimes k}, k(P') \oplus k(Q'))\end{aligned}$$

When viewed as an element of $\mathrm{Hom}(\omega_X^{\otimes k}, k(P') \oplus k(Q'))$, $\phi$ corresponds to the composition of

$$r_{P',Q'} : \omega_X^{\otimes k} \longrightarrow \omega_{X_{P'}}^{\otimes k} \oplus \omega_{X_{Q'}}^{\otimes k}$$

and the projection map

$$\omega_{X_{P'}}^{\otimes k} \oplus \omega_{X_{Q'}}^{\otimes k} \twoheadrightarrow k(P') \oplus k(Q')$$

Indeed, as $\omega_X^{\otimes k}$ is an invertible sheaf on $X$, $\mathrm{Hom}(\omega_X^{\otimes k}, k(x)) = k$ for all closed points $x \in X$, and thus, up to scaling, there is only one non-trivial homomorphism $\omega_X^{\otimes k} \longrightarrow k(P')$ and $\omega_X^{\otimes k} \longrightarrow k(Q')$ which means that $\phi$ must correspond to the above composition as it is non-trivial on both components of the direct sum. Now the commutativity of the diagram

$$\begin{array}{ccc}
\Gamma(Y,\omega_Y^{\otimes k}) & \xrightarrow{r_{P,Q}(Y)} & \Gamma(Y,k(P)\oplus k(Q)) \\
\downarrow = & & \downarrow = \\
\mathrm{Hom}(\mathcal{O}_Y,\omega_Y^{\otimes k}) & \xrightarrow{r_{P,Q}(Y)\circ} & \mathrm{Hom}(\mathcal{O}_Y,k(P)\oplus k(Q)) \\
\downarrow = & & \downarrow = \\
\mathrm{Hom}(\mathcal{O}_X,\omega_X^{\otimes k}) & \xrightarrow{r_{P',Q'}(X)\circ} & \mathrm{Hom}(\mathcal{O}_X,k(P')\oplus k(Q')) \\
\downarrow = & & \downarrow = \\
\Gamma(X,\omega_X^{\otimes k}) & \xrightarrow{r_{P',Q'}(X)} & \Gamma(X,k(P')\oplus k(Q'))
\end{array}$$

together with the fact that $r_{P',Q'}(X)$ is surjective (as $\omega_X^{\otimes k}$ is very ample) implies that $r_{P,Q}(Y)$ is surjective as well and thus $\omega_Y^{\otimes k}$ separates points.

Next, we want to show that $\omega_Y^{\otimes k}$ separates tangent vectors. Thus, let $y \in Y$ and $f \in (\mathfrak{m}_y/\mathfrak{m}_y^2)^\vee$ be a tangent vector. Then by chapter 1, lemma 1.18, $f$ corresponds to a closed subscheme $Z_y$ concentrated in $y$ with structure sheaf isomorphic to $k[\epsilon]/(\epsilon^2)$.Thus, we have in particular an exact sequence

$$0 \longrightarrow k \xrightarrow{\cdot\epsilon} k[\epsilon]/(\epsilon^2) \longrightarrow k \longrightarrow 0$$

which corresponds to a non-trivial extension

$$\begin{aligned}
e_Z \in \mathrm{Ext}^1(k(y),k(y)) &= \mathrm{Hom}(k(y),k(y)[1]) \\
&= \mathrm{Hom}(F(k(x)),F(k(x))[1]) \\
&= \mathrm{Ext}^1(k(x),k(x))
\end{aligned}$$

for some closed point $x \in X$. Therefore, $e_Z$ gives rise to an exact sequence

$$0 \longrightarrow k(x) \longrightarrow \mathcal{O}_{Z_x} \longrightarrow k(x) \longrightarrow 0$$

which defines a subscheme $Z_x$ of length two concentrated in $x \in X$, i.e. $\mathcal{O}_{Z_x}$ is isomorphic to $k[\epsilon]/(\epsilon^2)$ and therefore corresponds to a tangent vector in $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$. Furthermore, we have $F(\mathcal{O}_{Z_x}) = \mathcal{O}_{Z_y}$ by construction and $\Gamma(X,\mathcal{O}_{Z_x}) = \Gamma(Y,\mathcal{O}_{Z_y})$ as

$$\begin{aligned}
\Gamma(X,\mathcal{O}_{Z_x}) &= \mathrm{Hom}(\mathcal{O}_X,\mathcal{O}_{Z_x}) \\
&= \mathrm{Hom}(F(\mathcal{O}_X),F(\mathcal{O}_{Z_x})) \\
&= \mathrm{Hom}(\mathcal{O}_Y,\mathcal{O}_{Z_y}) \\
&= \Gamma(Y,\mathcal{O}_{Z_y})
\end{aligned}$$

As $\omega_X^{\otimes k}$ separates tangent vectors, we know that the restriction map $\omega_X^{\otimes k} \longrightarrow \mathcal{O}_{Z_x}$ induces a surjection $\Gamma(X,\omega_X^{\otimes k}) \longrightarrow \Gamma(X,\mathcal{O}_{Z_x})$ by chapter 1, lemma 1.19. Now, one can check that

$$F(\varphi : \omega_X^{\otimes k} \longrightarrow \mathcal{O}_{Z_x}) \cong \varphi' : \omega_Y^{\otimes k} \longrightarrow \mathcal{O}_{Z_y}$$

where the maps on both sides are induced by the restriction maps and the isomorphism means equality up to composition by an automorphisms of $\mathcal{O}_{Z_y}$. In order to see this, first notice that we have an isomorphism of $k$-vector spaces $\mathrm{Hom}(\omega_X^{\otimes k},\mathcal{O}_{Z_x}) \cong$

$\mathrm{Hom}(\omega_Y^{\otimes k}, \mathcal{O}_{Z_y})$ induced by $F$. Let $f_X^{\#} : \mathcal{O}_X \longrightarrow i_*(\mathcal{O}_{Z_x})$ and $f_Y^{\#} : \mathcal{O}_Y \longrightarrow i_*(\mathcal{O}_{Z_y})$ be the natural morphisms. Notice that any morphism of $\mathcal{O}_V$-modules $f : \omega_V^{\otimes k} \longrightarrow \mathcal{O}_{Z_v}$ factors as $f' \circ \pi_v$, for $V = X, Y$ and $v = x, y$ respectively, where $\pi_v$ maps a section $s \in \omega_V^{\otimes k}(U)$ to the germ $s_v \in \omega_{V,v}^{\otimes k} \cong \mathcal{O}_{V,v}$ if $v \in U$ and to 0 if $v \notin U$. Here, $f' : \mathcal{O}_{V,v} \longrightarrow i_*(\mathcal{O}_{Z_v})_v$ is a map of $\mathcal{O}_{V,v}$-modules. For all $s \in \mathcal{O}_{V,v}$, we have $f'(s) = f'(s \cdot 1) = f_V^{\#}(s) f'(1) = (f_V^{\#}(s))(c + d\epsilon)$ for $c, d \in k$. Thus, any morphism of $\mathcal{O}_V$-modules $f : \omega_V^{\otimes k} \longrightarrow \mathcal{O}_{Z_v}$ is of the form $s \mapsto (f_V^{\#}(\pi_v(s)))(c + d\epsilon)$ for $c, d \in k$ and we see that it is induced by the restriction map followed by an endomorphism of $\mathcal{O}_{Z_v}$. We want to prove that $F(\varphi)$ is of the form $s \mapsto (f_V^{\#}(\pi_v(s)))(c + d\epsilon)$ with $c \neq 0$. This must be true: $\varphi$ has the property that when composed with any non-trivial endomorphism of $\mathcal{O}_{Z_x}$, the composition remains non-trivial (we can check this on the stalk $\mathcal{O}_{X,x}$). As $F$ maps non-trivial morphisms to non-trivial morphisms and induces isomorphisms $\mathrm{Hom}(\omega_X^{\otimes k}, \mathcal{O}_{Z_x}) \cong \mathrm{Hom}(\omega_Y^{\otimes k}, \mathcal{O}_{Z_y})$ and $\mathrm{End}(\mathcal{O}_{Z_x}) \cong \mathrm{End}(\mathcal{O}_{Z_y})$, we see that $F(\varphi)$ must have the same property: assume we have found a non-trivial element $t \in \mathrm{End}(\mathcal{O}_{Z_y})$ such that $t \circ F(\varphi) = 0$, then we will find a non-trivial $t' \in \mathrm{End}(\mathcal{O}_{Z_x})$ with $F(t') = t$ and $t' \circ \varphi = 0$ which is a contradiction. But if we have $c = 0$, then $F(\varphi)$ will be of the form $s \mapsto (f_V^{\#}(\pi_v(s)))(d\epsilon)$ and we see that its composition with the non-trivial endomorphism "multiplication by $\epsilon$" is trivial, which is a contradiction and thus $c \neq 0$. But if $c \neq 0$, then multiplication by $c + d\epsilon$ is an automorphism of $\mathcal{O}_{Z_y}$ (with inverse multiplication by $\frac{1}{c} - \frac{d}{c^2}\epsilon$). This proves $F(\varphi : \omega_X^{\otimes k} \longrightarrow \mathcal{O}_{Z_x}) \cong \varphi' : \omega_Y^{\otimes k} \longrightarrow \mathcal{O}_{Z_y}$ and also immediately implies the following: if $\varphi$ induces a surjection $\Gamma(X, \omega_X^{\otimes k}) \longrightarrow \Gamma(X, \mathcal{O}_{Z_x})$, then so must $\varphi' : \Gamma(Y, \omega_Y^{\otimes k}) \longrightarrow \Gamma(Y, \mathcal{O}_{Z_y})$. Thus $\omega_Y^{\otimes k}$ separates tangent vectors by lemma 1.19 from chapter 1 which yields that it is very ample and thus we have finished the proof.

□

**Remark:** Throughout the proof of this main theorem, we have often assumed $X$ to have ample canonical bundle. However, one can easily check that it also suffices for $X$ to have ample anti-canonical bundle, i.e. to require that $\omega_X^{-1}$ is ample (we only used that there exists an integer $d$ - possibly negative - such that $\omega_X^d$ is very ample). Under this assumption, we can use the same proof with only very minor modifications. Also, notice that we never used that our equivalence $F : D^b(X) \longrightarrow D^b(Y)$ sends exact triangles to exact triangles - it is enough to require that $F$ commutes with the shift functors of $D^b(X)$ and $D^b(Y)$. A functor with this property is called *graded*.

CHAPTER 3

# Reconstructing Chow groups

As we saw in the previous chapter, the proof of Bondal and Orlov's theorem made heavy use of the fact that we can actually transfer the geometry of $X$ to the derived category $D^b(X)$, by identifying closed points and invertible sheaves on $X$. The latter corresponds to rational equivalence classes of closed subvarieties of codimension 1 when considered as divisors on $X$. Thus, a natural question to consider is whether we can reconstruct more of the geometry of $X$ from its derived category. In view of the previous results, a good candidate could be the additive structure of the Chow ring of $X$ (which will be referred to as the *Chow group of* $X$ in the following), which would require us to identify closed subvarieties of codimension $r$ for any $0 \leq r \leq n$ via rational equivalence. To answer this question, we need to pass to the slightly more general setting of tensor-triangulated categories: so far, when we looked at the category $D^b(X)$ and we considered it as a triangulated category. In fact, we never even used the triangles in $D^b(X)$ but only worked with the graded structure of the category, i.e. we used its shift functor. In the following, the triangles in $D^b(X)$ will play a role in the sense that we need them to define thick subcategories. But we need more: note that for $X$ smooth projective, $D^b(X)$ carries the structure of a left-derived tensor product, an exact bi-functor $D^b(X) \times D^b(X) \longrightarrow D^b(X)$. With this additional structure, we can use the work of P. Balmer (cf. [**Bal02**], [**Bal05**], [**Bal09**]) to achieve our goal.

## 1. Perfect complexes

Let $X$ be an algebraic variety over an algebraically closed field and let $\mathcal{F}^\bullet$ be a complex of $\mathcal{O}_X$-modules on $X$.

**Definition 1.1.** *$\mathcal{F}^\bullet$ is called* strictly perfect *if $X$ can be covered by opens $U_i, i \in I$ such that $\mathcal{F}^\bullet|_{U_i}$ is a bounded complex of locally free sheaves of finite rank for all $i$. $\mathcal{F}^\bullet$ is called* perfect *if it is locally quasi-isomorphic to a strictly perfect complex.*

**Definition 1.2.** *The category $D^{perf}(X)$ of perfect complexes on $X$ is defined as the full subcategory of perfect complexes of $D^b(X)$.*

**Remark:** We could also define $D^{\mathrm{perf}}(X)$ as the full subcategory of perfect complexes of $D^b(\mathrm{Mod}(X))$. Indeed, by [**Huy06**, Corollary 3.4], we have that the natural inclusion functor $D^b(\mathrm{Qcoh}(X)) \hookrightarrow D^b_{\mathrm{qcoh}}(\mathrm{Mod}(X))$ is an equivalence, where $D^b_{\mathrm{qcoh}}(\mathrm{Mod}(X))$ is the full subcategory of $D^b(\mathrm{Mod}(X))$ of complexes with quasi-coherent cohomology. Furthermore, [**Huy06**, Proposition 3.5] says that the natural inclusion functor $D^b(X) \hookrightarrow D^b_{\mathrm{coh}}(\mathrm{Qcoh}(X))$ is an equivalence, where $D^b_{\mathrm{coh}}(\mathrm{Qcoh}(X))$ is the full subcategory of $D^b(\mathrm{Qcoh}(X))$ of complexes with coherent cohomology. These results say that the complexes in $D^b(\mathrm{Mod}(X))$ with coherent cohomology are exactly those coming from the natural inclusion $D^b(X) \hookrightarrow$

$D^b(\mathrm{Mod}(X))$. Now, let $a$ be an element of the full subcategory of perfect complexes of $D^b(\mathrm{Mod}(X))$. As quasi-isomorphisms preserve cohomology and locally free sheaves are coherent, this means that $a$ has coherent cohomology, which implies that it is an element of $D^b(X)$. Thus it is also an element of $D^{\mathrm{perf}}(X)$. On the other hand, if $a \in D^{\mathrm{perf}}(X)$, then $a$ is automatically en element of the full subcategory of perfect complexes of $D^b(\mathrm{Mod}(X))$ as we have the natural functor $D^b(X) \hookrightarrow D^b(\mathrm{Mod}(X)$

If $X$ is smooth then we have the following fact (cf. [**Huy06**, Proposition 3.26]):

**Proposition 1.3.** *If $X$ is regular, then any $\mathcal{F}^\bullet \in D^b(X)$ is isomorphic to a bounded complex of locally free sheaves in $D^b(X)$.*

This allows for the following conclusion which connects the theory for $D^b(X)$ we have considered so far with the theory for $D^{\mathrm{perf}}(X)$ we will be looking at:

**Corollary 1.4.** *If $X$ is regular, then $D^{perf}(X) = D^b(X)$.*

Proof. As we already have $D^{\mathrm{perf}}(X) \subset D^b(X)$ as a full subcategory, we only need to prove $D^{\mathrm{perf}}(X) \supset D^b(X)$. Let $a \in D^b(X)$, then by proposition 1.3 $a$ is isomorphic to a bounded complex of locally free sheaves in $D^b(X)$. This means that it is quasi-isomorphic to a strictly perfect complex which implies that it is perfect, which concludes the proof. □

**Remark:** In this case, this shows that $D^{\mathrm{perf}}(X)$ is triangulated, as $D^b(X)$ is. However, this is true more generally as we will see later on.

**Notation:** In the following we will be mostly dealing with smooth projective varieties. For them, *we will use the notation $D^{perf}(X)$ and $D^b(X)$ interchangeably*, which is justified by the preceding corollary.

## 2. Balmer's prime spectrum

In this section, we define the setting we'll be working in for the rest of this thesis and present the results by P. Balmer that will be fundamental for our considerations.

**Definition 2.1.** *Let $K$ be a triangulated category. A subcategory $L \subset K$ is called* thick *if it is a full triangulated subcategory such that $P \oplus Q \in A$ with $P, Q \in A$ forces $P \in A$ and $Q \in A$.*

**Definition 2.2.** *A* tensor-triangulated category *is a triple $(K, \otimes, \mathbb{I})$, where $K$ is a triangulated category, $\otimes$ is a functor*

$$\otimes : K \times K \longrightarrow K$$

*that induces a symmetric monoidal structure on $K$ and that is exact in each variable. Furthermore, $\mathbb{I}$ is a unit with respect to $\otimes$, i.e. for all $A \in K$ we have natural isomorphisms $A \otimes \mathbb{I} \cong \mathbb{I} \otimes A \cong A$.*

**Notation:** Mostly, we will simply write $K$ for a tensor-triangulated category and leave out $\otimes$ and $\mathbb{I}$ when there is no danger of confusion.

**Example:** For a smooth projective variety $X$, $D^b(X)$ is tensor-triangulated category with the usual left-derived tensor product $\otimes^L$ and unit $\mathcal{O}_X$. Recall that for two complexes

$\mathcal{E}^\bullet, \mathcal{F}^\bullet$, we define the tensor product $\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet$ by

$$(\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet)^i = \bigoplus_{p+q=i} \mathcal{E}^p \otimes \mathcal{F}^q \text{ with } d = d_{\mathcal{F}} \otimes 1 + (-1)^i d_{\mathcal{E}}$$

For all details in this special case, we refer the reader to [**Huy06**, Chapter 3.3]. Note however that we really need the regularity of $X$ to make sure that $D^b(X) = D^{\mathrm{perf}}(X)$, which makes the definition of $\otimes^L$ possible.

**Definition 2.3.** *Let $(K, \otimes, \mathbb{I})$ be a tensor-triangulated category. A thick subcategory $L \subset K$ is called* $\otimes$-thick *if for all $P \in L, Q \in K$ we have $P \otimes Q \in L$.*

**Example:** Let $(K, \otimes, \mathbb{I}) = (D^{\mathrm{perf}}(X), \otimes^L, \mathcal{O}_X)$. Then, for some subset $Y \subset X$ define the full subcategory $K_Y := \{a \in D^{\mathrm{perf}}(X) | \mathrm{supp}(a) \subset Y\}$. This is a $\otimes$-thick subcategory (cf. [**Bal05**, Lemma 3.4]).

**Construction:** Following [**Bal05**, Balmer], we can associate to a tensor-triangulated category $(K, \otimes, \mathbb{I})$ a topological space $\mathrm{Spc}(K)$ and a sheaf of commutative rings $\mathcal{O}_K$ on $\mathrm{Spc}(K)$. In the case that $X$ is a topologically noetherian scheme he shows that $(X, \mathcal{O}_X) \cong (\mathrm{Spc}(D^{\mathrm{perf}}(X), \mathcal{O}_{D^{\mathrm{perf}}(X)})$. In the following two sections, we will take a closer look at this construction.

**2.1. The associated topological space.** In this section, $K = (K, \otimes, \mathbb{I})$ will always denote a tensor-triangulated category with tensor product $\otimes$ and unit $\mathbb{I}$.

**Definition 2.4.** *Let $A \subset K$ be a $\otimes$-thick subcategory. $A$ is called* prime *if for every two objects $a, b \in K$ with $a \otimes b \in A$ we either have $a \in A$ or $b \in A$. The* radical *of $A$ is defined as $\sqrt{A} := \{a \in K | a^{\otimes n} \in A$ for some $n \in \mathbb{N}\}$. $A$ is called* radical *if $\sqrt{A} = A$.*

**Example:** Let $X$ be a smooth projective variety, $Z \subset X$ a closed subset and $x \in X$ a point. Consider the $\otimes^L$-thick subcategory $D^{\mathrm{perf}}_Z(X)$ of complexes $a$ with $\mathrm{supp}(a) \subset Z$. Then $D^{\mathrm{perf}}_Z(X)$ is radical: indeed, as $\mathrm{supp}(a^{\otimes^L n}) = \mathrm{supp}(a)$ for all $n \in \mathbb{N}$ we see that $a^{\otimes^L n} \in D^{\mathrm{perf}}_Z(X)$ implies $a \in D^{\mathrm{perf}}_Z(X)$. Next, let $V_x := \{a \in D^{\mathrm{perf}}(X) | a_x \cong 0 \in D^{\mathrm{perf}}(\mathcal{O}_{X,x})\}$. Then $V_x$ is prime: indeed, let $a \otimes^L b \in V_x$, then $(a \otimes^L b)_x \cong a_x \otimes^L b_x \cong 0 \in D^{\mathrm{perf}}(\mathcal{O}_{X,x})$. As the tensor product of two finitely generated local modules is zero iff one of the modules is zero (cf. [Serre: Local Algebra, Corollary I.2.2]), this proves the claim.

Now we define the spectrum $\mathrm{Spc}(K)$ as a set:

**Definition 2.5.** *We define $Spc(K)$ as the set of all prime subcategories of $K$. We define a topology on $Spc(K)$ by the basis of open sets $U(a) := \{P \in Spc(K) | a \in P\}$ for all $a \in K$. The closed complement of $U(a)$ is denoted by $supp(a) := Spc(K) \setminus U(a)$.*

This is indeed a topology basis, which is proved in [**Bal05**, Remark 2.7]. In particular we have the following nice lemma:

**Lemma 2.6.** *Let $a \in K$, then*

$$U(a \oplus b) = U(a) \cap U(b) \text{ and } U(a \otimes b) = U(a) \cup U(b)$$

PROOF. cf. [**Bal05**, Lemma 2.6] □

This allows for the following definition:

**Definition 2.7.** *The* dimension *of $K$, denoted by $dim(K)$ is defined as the dimension of $Spc(K)$ as a topological space.*

A useful property of the spectrum of a tensor-triangulated category is the following:

**Proposition 2.8.** *Every non-empty, closed irreducible subset of $Spc(K)$ has a unique generic point.*

PROOF. cf. [**Bal05**, Proposition 2.18] □

**Remark:** If $\mathrm{Spc}(K)$ is a noetherian topological space, then this means that $\mathrm{Spc}(K)$ is a Zariski space in the sense of [**Har77**, II.3 Exercise 3.17]. Balmer shows that $\mathrm{Spc}(K)$ is noetherian (and thus a Zariski space) if and only if any closed subset of $\mathrm{Spc}(K)$ is the support of an object of $K$ (cf. [**Bal05**, Corollary 2.17]).

The most important application of Balmer's theory for this thesis are the following two theorems:

**Theorem 2.9.** *Let $\mathfrak{S}$ be the set of those subsets $Y \subset Spc(K)$ of the form $Y = \bigcup_{i \in I} Y_i$ for closed subsets $Y_i$ of $Spc(K)$ with $Y_i$ quasi-compact for all $i \in I$. Let $\mathfrak{R}$ be the set of radical $\otimes$-thick subcategories of $K$. Then there is an order-preserving bijection $\mathfrak{S} \xrightarrow{\sim} \mathfrak{R}$ given by*

$$Y \mapsto K_Y := \{a \in K | supp(a) \subset Y\}$$

*with inverse*

$$J \mapsto supp(J) := \bigcup_{a \in J} supp(a)$$

PROOF. cf. [**Bal05**, Theorem 4.10] □

**Remark:** The statement of this theorem becomes easier, if we assume that all $\otimes$-thick subcategories are radical and that all closed subsets are of the form $\mathrm{supp}(a)$ for some $a \in K$. Then it reads as a more or less direct generalization of Thomason's classification of thick subcategories, as stated in [**Bal02**, Corollary 2.8].

**Theorem 2.10.** *Let $X$ be a smooth projective variety. There is a homeomorphism*

$$\begin{array}{ccc} X & \xrightarrow{\sim} & Spc(D^{perf}(X)) \\ x & \mapsto & \{a \in D^{perf}(X) | a_x \cong 0 \in D^{perf}(\mathcal{O}_{X,x})\} \end{array}$$

*Moreover, for any perfect complex $a \in D^{perf}(X)$, the closed subset $supp(a)$ in $X$ corresponds via $f$ to the closed subset $supp(a)$ in $Spc(D^{perf}(X))$.*

PROOF. cf. [**Bal05**, Corollary 5.6] □

**Remark:** We see that all closed subsets of $\mathrm{Spc}(D^{\mathrm{perf}}(X))$ are of the form $\mathrm{supp}(a)$ for some $a \in D^{\mathrm{perf}}(X)$: let $Z \subset \mathrm{Spc}(D^{\mathrm{perf}}(X))$ be closed, then $f^{-1}(Z) \subset X$ is also closed. Then by [**Rou09**, Lemma 4.7], there exists an $a \in D^{\mathrm{perf}}(X)$ with $\mathrm{supp}(a) = f^{-1}(Z)$. By the preceding theorem, this implies that $Z = \mathrm{supp}(a)$.

**2.2. The associated sheaf of rings.** Before we take a closer look at the construction of the sheaf of rings $\mathcal{O}_K$, we first need to state an existence theorem for quotient categories. We stick to the treatment in [**Nee01**, Chapter 2.1]

**Definition 2.11.** *Let $\mathcal{D}, \mathcal{T}$ be triangulated categories and $F : \mathcal{D} \longrightarrow \mathcal{T}$ be a triangulated functor. The* kernel of $F$ *is defined as the full subcategory*

$$ker(F) := \{a \in \mathcal{D} | F(a) \cong 0\}.$$

The following theorem states that we can "divide out" by thick subcategories:

**Theorem 2.12.** *Let $\mathcal{D}$ be a triangulated category and $\mathcal{C} \subset \mathcal{D}$ be a thick subcategory. Then there exists a triangulated category $\mathcal{D}/\mathcal{C}$ and a triangulated functor $F_{un} : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ so that $ker(F_{un}) = \mathcal{C}$ and $F_{un}$ is universal with this property: if $F : \mathcal{D} \longrightarrow \mathcal{T}$ is a triangulated functor whose kernel contains $\mathcal{C}$, then it factors uniquely as*

$$\mathcal{D} \xrightarrow{F_{un}} \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{T}$$

Proof. This is [**Nee01**, Theorem 2.1.8]. □

**Remark:** Actually, we don't need $\mathcal{C}$ to be thick, however, if we drop this assumption then we will only have $\mathcal{C} \subset \ker(F)$. The objects of $\mathcal{D}/\mathcal{C}$ are just the objects of $\mathcal{D}$, and the functor $F_{\text{un}}$ is the identity on objects (cf. [**Nee01**, Remark 2.1.10]). The essence of the quotient category lies in the construction of the morphisms of $\mathcal{D}/\mathcal{C}$.

**Definition 2.13.** *The category $\mathcal{D}/\mathcal{C}$ is called the* Verdier quotient of $\mathcal{D}$ by $\mathcal{C}$ *and the functor $F_{un}$ is called the* Verdier localization map*.*

**Construction of the sheaf of rings:** Let us now give an overview on how to define the sheaf of rings $\mathcal{O}_K$ on $\text{Spc}(K)$, following [**Bal05**, Definition 6.1] and [**Bal09**, Construction 6.1]. Let $(K, \otimes, \mathbb{I})$ be a tensor-triangulated category and $U \subset \text{Spc}(K)$ an open subset. Let $Z := \text{Spc}(K) \setminus U$ be its closed complement and let $K_Z := \{a \in K | \text{supp}(a) \subset Z\}$ (this is a $\otimes$-thick subcategory of $K$, cf. [**Bal05**, Lemma 3.4]). Define a sheaf $\mathcal{O}_K$ on $X$ as the sheafification of the presheaf

$$\mathcal{O}'_K : U \mapsto \text{End}_{(K/K_Z)}(\mathbb{I}_U)$$

where the unit $\mathbb{I}_U \in K/K_Z$ is the image of the unit $\mathbb{I}$ of $K$ via the localization $K \mapsto K/K_Z$. From the proof of [**Bal02**, Proposition 5.3] it follows that we have a presheaf

$$U \mapsto \widetilde{K/K_Z},$$

where $\widetilde{K/K_Z}$ is the idempotent completion of $K/K_Z$. The restriction maps of this presheaf induce the restriction maps of $\mathcal{O}'_K$: for $V_1 \subset V_2$, we have a tensor-triangulated functor $\widetilde{K/K_{Z_2}} \longrightarrow \widetilde{K/K_{Z_1}}$ which induces maps $\text{End}_{\widetilde{K/K_{Z_2}}}(\mathbb{I}_{V_2}) \longrightarrow \text{End}_{\widetilde{K/K_{Z_1}}}(\mathbb{I}_{V_1})$. As the canonical functor that sends a category to its idempotent completion is fully faithful, we have $\text{End}_{\widetilde{K/K_{Z_2}}}(\mathbb{I}_{V_2}) = \text{End}_{K/K_{Z_2}}(\mathbb{I}_{V_2})$ and $\text{End}_{\widetilde{K/K_{Z_1}}}(\mathbb{I}_{V_2}) = \text{End}_{K/K_{Z_1}}(\mathbb{I}_{V_1})$, which means that we get restriction maps $\mathcal{O}'_K(V_2) \longrightarrow \mathcal{O}'_K(V_1)$. One can check that these turn $\mathcal{O}'_K$ into a presheaf.

If $K$ is $k$-linear for some field $k$, then $\mathcal{O}'_K(\text{Spc}(K)) = \text{End}_K(\mathbb{I})$ is a $k$-vector space and in particular we can embed $k \hookrightarrow \mathcal{O}'_K(\text{Spc}(K))$ by sending $a \mapsto a \cdot \text{id}$ for any $a \in k$

(This is indeed a ring homomorphism: additivity is clear as $\mathrm{End}_K(\mathbb{I})$ is a $k$-vector space. Multiplicativity follows from the fact that the composition of maps is bilinear). As we have a homomorphism $\mathcal{O}'_K \longrightarrow \mathcal{O}_K$, this makes $\mathcal{O}_K(\mathrm{Spc}(K))$ into a $k$-algebra. Now we have the following:

**Definition 2.14.** *Let $(K, \otimes, \mathbb{I})$ be a tensor-triangulated category. Then define the ringed spaces*

$$\mathit{Spec}(K) := (\mathit{Spc}(K), \mathcal{O}_K)$$

**Theorem 2.15.** *Let $(K, \otimes, \mathbb{I})$ be a tensor-triangulated category such that every $\otimes$-thick subcategory is radical. Then Spec$(K)$ is a locally ringed space.*

Proof. An announcement of the proof can be found in [**Bal05**, Remark 6.4], the proof itself in [**Bal09**, Corollary 6.6]. □

**Remark:** According to [**Bal05**, Remark 6.4] it is an open question when $\mathrm{Spec}(K)$ is actually a scheme. Therefore, we are forced to work with locally ringed spaces in general and the analogies we can produce between schemes and tensor-triangulated categories depend on how much theory for schemes is also valid for locally ringed spaces.

If we take $K$ to be $k$-linear, we can make the following statement:

**Lemma 2.16.** *Let $(K, \otimes, \mathbb{I})$ be a tensor-triangulated category that is $k$-linear and such that every $\otimes$-thick subcategory is radical. Then there is a morphism of locally ringed spaces Spec$(K) \longrightarrow$ Spec$(k)$.*

Proof. This is immediate from the fact that $k$-linearity of $K$ implies that $\mathcal{O}_K(\mathrm{Spc}(K))$ is a $k$-algebra. Indeed, a morphism of locally ringed spaces consists of a pair $(f, f^{\#})$, where $f$ is a continuous map $\mathrm{Spec}(K) \longrightarrow \mathrm{Spec}(k)$ and $f^{\#}$ is a local homomorphism of sheaves $\mathcal{O}_{\mathrm{Spec}(k)} \longrightarrow f_*(\mathcal{O}_K)$. Now, $f$ is simply given by the projection $\mathrm{Spc}(K) \longrightarrow P$, where $P$ is the single point of $\mathrm{Spec}(k)$. To give a map of sheaves $\mathcal{O}_{\mathrm{Spec}(k)} \longrightarrow f_*(\mathcal{O}_K)$, it suffices to give a ring homomorphism $f' : k \longrightarrow \mathcal{O}_K(\mathrm{Spc}(K))$ which exists as $\mathcal{O}_K(\mathrm{Spc}(K))$ is a $k$-algebra. To check that this homomorphism is local, simply remark that $f'$ sends invertible elements to invertible elements, the unique maximal ideal of any local ring of $\mathrm{Spec}(K)$ consists of exactly the non-units of the ring and that 0 is the only non-unit of $k$ and its unique maximal ideal at the same time. □

**Remark:** The importance of this lemma arises from the fact that later on, we want to take fibred products over $k$: for a smooth projective variety $X$ over $k$ we need to define $X \times_k \mathrm{Spec}(K)$, which requires the existence of the morphism described in the previous lemma.

Now we can state Balmer's full reconstruction theorem:

**Theorem 2.17.** *For a topologically noetherian scheme $X$, we have that*

$$\mathit{Spec}(D^{perf}(X)) \cong X$$

Proof. cf. [**Bal05**, Theorem 6.3] □

**Remark:** The following result by Balmer (cf. [**Bal02**, Theorem 2.13]) roughly explains how the sheaf of rings are isomorphic: for a topologically noetherian scheme $X$, $U \subset X$ an open subscheme with closed complement $Z$, we have an equivalence of categories

$$\widetilde{D^{\mathrm{perf}}(X)/D_Z^{\mathrm{perf}}}(X) \longrightarrow D^{\mathrm{perf}}(U)$$

where $\widetilde{D^{\mathrm{perf}}(X)/D_Z^{\mathrm{perf}}}(X)$ is the idempotent completion of $D^{\mathrm{perf}}(X)/D_Z^{\mathrm{perf}}(X)$ and $D_Z^{\mathrm{perf}}(X)$ is the thick subcategory of complexes with cohomological support in $Z$. From this, one can deduce that $\mathrm{End}(\mathbb{I}_U) = \mathrm{End}(\mathcal{O}_X(U)) \cong \mathcal{O}_X(U)$.

Before we start applying the concepts we introduced, we want to show a quick example of how strong the machinery developed by Balmer is. It implies a variant of Bondal and Orlov's theorem:

**Theorem 2.18.** *Let $X, Y$ be smooth projective varieties and assume there is a tensor-triangulated equivalence $F : D^b(X) \longrightarrow D^b(Y)$, i.e. an exact equivalence that commutes with the tensor products on $D^b(X)$ and $D^b(Y)$. Then $X \cong Y$.*

Proof. This is [**Bal02**, Theorem 9.7]. Note that for the proof found there, one uses a slightly different definition of $\mathrm{Spec}(K)$. □

**Remark:** Here, we don't require $X$ to have ample (anti-)canonical bundle, so in this sense the theorem is a generalization. However there is an obvious trade-off: we do require the existence of a tensor-triangulated equivalence, which is much more than just the graded equivalence we needed for the proof of Bondal and Orlov's result. The theorem does *not* hold if the equivalence is not tensor-triangulated (for counterexamples cf. [**Orl96**]).

## 3. Reminder: rational equivalence

Our next goal is to transport the Chow group of a variety to the level of tensor-triangulated categories. Therefore we need the notion of rational equivalence. Recall the following definitions in the geometric setting:

**Definition 3.1.** *Let $X$ be a scheme. An* algebraic $k$-cycle *on $X$ is an element of the free abelian group generated by the closed irreducible subvarieties of dimension $k$ in $X$, i.e. an element of the form*

$$\sum_i n_i[V_i]$$

*with $V_i \subset X, dim(V_i) = k, n_i \in \mathbb{Z}$ for all $i$. Denote the free abelian group generated by the closed irreducible subvarieties of dimension $k$ in $X$ by $Z_k(X)$.*

**Definition 3.2.** *Let $f : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then $f$ is called* dominant*, if its image is dense in $Y$.*

**Definition 3.3.** *Let $V$ be a scheme with irreducible components $V_1, \ldots, V_t$. The* geometric multiplicity of $V_i$ in $V$ *is defined as*

$$m_i := length(\mathcal{O}_{V;V_i}),$$

*the length of the local ring of* $V_i$. *The* fundamental cycle of $V$ *is defined as*

$$[V] := \sum_i m_i[V_i]$$

*and is interpreted as an element of* $\bigoplus_k Z_k(X)$.

Now we come to the definition of rational equivalence and Chow groups. We follow [**Ful98**].

**Definition 3.4.** *Let* $X$ *be a smooth projective variety and let* $\alpha \in Z_k(X)$. $\alpha$ *is called* rationally equivalent to zero *if there exist* $k+1$*-dimensional subvarieties* $V_1, \dots, V_t \subset X \times \mathbb{P}^1$ *such that the natural projections* $\pi_i : V_i \longrightarrow \mathbb{P}^1$ *are dominant morphisms and we have that*

$$\alpha = \sum_{i=1}^{t} [V_i(0)] - [V_i(\infty)]$$

*where* $V_i(P)$ *is the projection of the scheme-theoretic fibre of* $\pi_i$ *over* $P$ *to* $X$. *Two cycles* $\beta, \gamma \in Z_k(X)$ *are called* rationally equivalent *if their difference is rationally equivalent to zero.*

**Remark:** In [**Ful98**] a definition of rational equivalence is given via divisors on subvarieties on $X$. It is shown to be equivalent to the above definition in [**Ful98**, Proposition 1.6]. Fulton also shows that the cycles rationally equivalent to zero form a subgroup of $Z_n$.

**Definition 3.5.** *Let* $X$ *be a smooth projective variety. For* $0 \leq n \leq dim(X)$, *let* $Z_0^n$ *be the subgroup of* $Z_n(X)$ *that consists of all cycles rationally equivalent to zero. Set* $A_n(X) = Z_n(X)/Z_0^n$ *and define the* Chow group of $X$ *as the group*

$$\bigoplus_{n=0}^{dim(X)} A_n$$

## 4. Chow groups

In this section, we will define Chow groups for certain tensor-triangulated categories.

(∗) In the following, $(K, \otimes, \mathbb{I})$ is assumed to be a tensor-triangulated category with the following properties:

- Every $\otimes$-thick subcategory of $K$ is radical. (This can, for example, be achieved by requiring $K$ to be rigid (cf. [**Bal09**, Definition 1.5 and Section 6])).
- $K$ is $k$-linear for some algebraically closed field $k$. Together with the preceding property, this implies that Spec($K$) is a locally ringed space over $k$ (cf. Lemma 2.16).
- $K$ is finite-dimensional (in the sense of definition 2.7).

- For any closed subset $Y \subset \mathrm{Spc}(K)$ we have that $Y = \mathrm{supp}(a)$ for some $a \in K$. This implies that $\mathrm{Spc}(K)$ is a noetherian topological space (cf. [**Bal05**, Corollary 2.17]). [1]

The main example for such a category is given by $D^b(X)$, for $X$ a smooth projective variety.

**Definition 4.1.** *Let $J$ be a (radical) $\otimes$-thick subcategory of $K$. Then by theorem 2.9, supp($J$) $= Y$ for some unique $Y \subset$ Spc($K$). $J$ is called* admissible *if $Y$ is closed and irreducible.*

**Definition 4.2.** *Let $(K, \otimes, \mathbb{I})$ be a tensor-triangulated category. Denote by $Z_n(K)$ the free abelian group on the admissible subcategories $J$ with dim(supp($J$)) $= n$. The elements of $Z_n(K)$ are called* $n$-cycles *in $K$.*

**Remark:** As $\mathrm{supp}(J) \subseteq \mathrm{Spc}(K)$, we must have $\dim(\mathrm{supp}(J)) \leq \dim(\mathrm{Spc}(K)) < \infty$ by assumption.

**Definition 4.3.** *Let $(X, \mathcal{O}_X)$ be a locally ringed space and $Y \subset X$ be a closed irreducible subset. The* subvariety associated to $Y$ *is the locally ringed space $(Y, (\mathcal{O}_X/I_Y)|_Y)$, where $I_Y \subset \mathcal{O}_X$ is the sheaf of ideals given by*

$$U \mapsto \{s \in \mathcal{O}_X(U) | s_P \in \mathfrak{m}_P \textit{ for all } P \in U \cap Y\}$$

*where $\mathfrak{m}_P$ is the maximal ideal of the local ring $\mathcal{O}_{X,P}$. We will also refer to the subvariety associated to $Y$ simply as the* subvariety $Y$.

**Remark:** $I_Y$ is indeed a sheaf of ideals: $I_Y(U)$ is equal to $\bigcap_{P \in U \cap Y} f_P^{-1}(\mathfrak{m}_P)$, where $f_P : \mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,P}$ is the natural ring homomorphism. As the (arbitrary) intersection of ideals is an ideal, $I_Y(U)$ is one. It is clear that $I_Y$ is compatible with the natural restriction maps.

**Example:** Let $X$ be a smooth projective variety and $Y \subset X$ a closed irreducible subset. Then the natural subvariety structure on $Y$ coincides with the subvariety associated to $Y$ defined above: indeed, the natural subvariety structure on $Y$ is given by the sheaf of ideals $J_Y \subset \mathcal{O}_X$, $J_Y(U) = \{s \in \mathcal{O}_X(U) | s \text{ vanishes on all } P \in U \cap Y\}$. But $s$ vanishing on all $P \in U \cap Y$ is equivalent to $s_P \in \mathfrak{m}_P$ for all $P \in U \cap Y$. This yields that $J_Y$ coincides with the sheaf of ideals $I_Y$ defined above, which proves the claim.

**Notation:** Let $J$ be an admissible subcategory. Then $J = K_Y$ and denote by $Z(J)$ the subvariety associated to the closed subset $Y \subset \mathrm{Spec}(K)$. For a cycle $\alpha = \sum_i n_i A_i$ define $Z(\alpha) = \sum_i n_i Z(A_i)$.

Before we start with defining rational equivalence on $Z_n(K)$, we generalize geometric multiplicities to ringed spaces

**Definition 4.4.** *Let $(X, \mathcal{O}_X)$ be a topologically noetherian, locally ringed space such that every closed irreducible subset has a generic point and denote by $X_i, i \in I$ the closed irreducible components of $X$. Then the* geometric multiplicity of $X_i$ in $X$ *is given by*

$$m_i := \textit{length}(\mathcal{O}_{X,P_i}),$$

[1] The condition seems a bit artificial, and it would be desirable to replace it by a simpler one on the underlying category $K$. If we also demand that the bijection of 2.9 simplifies to $Y$ being specialization-closed subsets, then the pair $(\mathrm{Spc}(K), \mathrm{supp})$ becomes a *classifying support data* on $K$ in the sense of [**Bal05**, Definition 5.1]

*where $\mathcal{O}_{X,P_i}$ is the local ring at the generic point $P_i$ of $X_i$. We denote by $[X]$ the formal sum $[X] := \sum_i m_i' V_i$, where the $V_i$ are the subvarieties associated to the underlying spaces of $X_i$ and*

$$m_i' = \begin{cases} m_i & \textit{if } \ m_i < \infty \\ 0 & \textit{if } \ m_i = \infty \end{cases}$$

**Remark 1:** As $X$ is topologically noetherian, the decomposition of $X$ into finitely many irreducible components exists and is unique.

**Remark 2:** In this definition, we deal with the problem of infinite geometric multiplicities in a way that is hardly satisfying. The $m_i$'s are finite if $\mathcal{O}_{X,P_i}$ is artinian (which implies that it is noetherian). It is easy to construct examples of locally ringed spaces such that the $m_i$'s are infinite (For example, just take any irreducible topological space and the constant sheaf associated to a local ring that is not artinian). However we have the following

**Conjecture 4.5.** *Let $K$ be a Hom-finite tensor-triangulated category and $X_i, i \in I$ irreducible components of Spec($K$). Then length($\mathcal{O}_{K,P_i}$) $< \infty$ where $\mathcal{O}_{K,P_i}$ is the local ring at the generic point $P_i$ of $X_i$.*

**Remark:** This conjecture is motivated by the fact that $D^{\mathrm{perf}}(X)$ is a Hom-finite tensor-triangulated category for which $\mathrm{length}(\mathcal{O}_{X,P_i}) < \infty$. The sheaf of rings $\mathcal{O}_K$ is defined via endomorphism rings of $K$ and thus we hope that Hom-finiteness is the right condition to ensure finite geometric multiplicities.

**Notation:** Let $f : X \longrightarrow Y$ be a morphism of locally ringed spaces over an algebraically closed field $k$ and let $P \in Y$. Then we denote by $X(P) := X \times_k \mathrm{Spec}(k(P))$ the fibre of $f$ over the point $P$.

Now we have the tools to start defining the Chow group of a tensor-triangulated category which satisfies the properties $(*)$.

**Definition 4.6.** *Let $(K, \otimes, \mathbb{I})$ be a tensor-triangulated category and let $\alpha \in Z_n(K)$. Then $\alpha$ is called* rationally equivalent to zero *if there are subvarieties $V_1, \ldots, V_t \subset$ Spec($K$) $\times_k \mathbb{P}^1_k$ of dimension $n+1$ such that the projections $\pi_i : V_i \longrightarrow \mathbb{P}^1_k$ are dominant and*

$$Z(\alpha) = \sum_{i=1}^{t} [V_i(0)] - [V_i(\infty)]$$

*Here, we interpret $V_i(P)$ for $P = 0, \infty$ as a subspace of Spec($K$) via the projection $p :$ Spec($K$) $\times_k \mathbb{P}^1_k \longrightarrow$ Spec($K$).*

In order to define the Chow group of a tensor-triangulated category with the properties $(*)$, we need to work "modulo rational equivalence". This is made possible by the following lemma:

**Lemma 4.7.** *Let $(K, \otimes, \mathbb{I})$ be a tensor-triangulated category. The cycles rationally equivalent to zero form a subgroup of $Z_n(K)$.*

Proof. Let $Z_0$ denote the subset of $Z_n(K)$ that contains all cycles rationally equivalent to zero. We need to prove that $Z_0$ is closed under addition and taking inverses. For the first property, let $Z(\alpha) = \sum_{i=1}^{t}[V_i(0)] - [V_i(\infty)]$ and $Z(\beta) = \sum_{i=1}^{t}[W_i(0)] - [W_i(\infty)]$ (with notations as in definition 4.6). Then

$$Z(\alpha + \beta) = Z(\alpha) + Z(\beta) = \sum_{\substack{i=1,\dots,t \\ A=V,W}} [A_i(0)] - [A_i(\infty)]$$

which proves that $\alpha + \beta$ is rationally equivalent to zero. Thus $Z_0$ is closed under addition and we still need to prove that we can also take inverses: thus, let $Z(\alpha) = \sum_{i=1}^{t}[V_i(0)] - [V_i(\infty)] \in Z_0$, then $Z(-\alpha) = -Z(\alpha) = \sum_{i=1}^{t}[V_i(\infty)] - [V_i(0)]$. Next, take the automorphism $\frac{1}{x}$ of $\mathbb{P}_k^1$ that sends $0 \mapsto \infty$ and $\infty \mapsto 0$. This automorphism induces an automorphism of ringed spaces on $\mathrm{Spec}(K) \times_k \mathbb{P}_k^1$, denote the images of the $V_i$ under this automorphism by $V_i'$ for all $i$. Now it follows from our construction that $V_i'(0) = V_i(\infty)$ and $V_i'(\infty) = V_i(0)$ and thus $\sum_{i=1}^{t}[V_i'(0)] - [V_i'(\infty)] = \sum_{i=1}^{t}[V_i(\infty)] - [V_i(0)] = Z(-\alpha)$. This proves that $-\alpha$ is rationally equivalent to zero which shows that $Z_0$ is a subgroup of $Z_n(K)$.

□

Now we are able to take quotients and define Chow groups on the category level.

**Definition 4.8.** *Let $K$ be a tensor-triangulated category. Denote by $A_n(K)$ the group $Z_n(K)$ modulo the subgroup of cycles rationally equivalent to zero. The* rational Chow group *of $K$ is defined as $A(K) := \bigoplus_n A_n(K)$.*

This definition is sensible when our categories come from projective varieties:

**Theorem 4.9.** *Let $X$ be a smooth projective variety. Then $A(D^b(X)) = A(X)$.*

Proof. By the classification of radical $\otimes$-thick subcategories (cf. Theorem 2.9), we have a bijection between $Z_n(D^b(X))$ and the free abelian group on the closed irreducible subvarieties of $\mathrm{Spec}(K) \cong X$ (cf. Theorem 2.17) of dimension $n$. As we showed earlier, the "new" definition of a subvariety (cf. Definition 4.3) coincides with the classical one in the case that $X$ is an algebraic variety and thus, it only remains to prove that rational equivalence in $Z_n(D^b(X)$ coincides with classical rational equivalence of algebraic cycles under this bijection: let $\alpha, \beta$ be algebraic $n$-cycles that are rationally equivalent, i.e. there are subvarieties $V_1, \dots, V_t \subset X \times_k \mathbb{P}_k^1$ of dimension $n+1$, dominant over $\mathbb{P}_k^1$ such that

$$\alpha - \beta = \sum_{i=1}^{t}[V_i(0)] - [V_i(\infty)]$$

Now, as we have $\mathrm{Spec}(D^b(X)) \cong X$ our bijection takes $\alpha, \beta$ to $n$-cycles $A, B$ in $Z_n(D^b(X))$ and the $V_i$ will also define subvarieties $W_1, \dots W_t$ in $\mathrm{Spec}(D^b(X)) \times_k \mathbb{P}_k^1$ of dimension $n+1$ by the isomorphism $\mathrm{Spec}(D^b(X)) \cong X$, which are dominant over $\mathbb{P}_k^1$. We must have $Z(A-B) = \sum_{i=1}^{t}[W_i(0)] - [W_i(\infty)]$ which proves that $A$ and $B$ are rationally equivalent as cycles in $Z_n(D^b(X))$. For the other implication, we can take cycles $A, B \in Z_n(D^b(X))$ and just reverse the argument we have used for the previous step to get that the algebraic cycles $Z(A), Z(B)$ in $Z_n(X)$ are rationally equivalent in the classical sense.

□

## 5. An outlook to intersection theory

The theory created in the previous chapters at hand, we can almost define a multiplicative structure on the Chow group of tensor-triangulated categories. The main obstruction that keeps us from doing this is that we do not know if Chow's moving lemma holds for our definition of rational equivalence. Let us explain the issue a bit more closely: the problem arises from the fact that we only want to intersect subspaces that "intersect properly" and therefore it is necessary to define an intersection product on the equivalence classes of the Chow group. In the algebro-geometric setting, Chow's moving lemma tells us that the equivalence class of the intersection of two classes of subvarieties does not depend on the choice of representatives and that we can always find representatives that "intersect properly". It is an open problem to prove a similar result for general ringed spaces or at least for $\mathrm{Spec}(K)$.

However it is possible to define an intersection product for two admissible subcategories of $K$ which do intersect properly (cf. Definition 5.1), with values in the free abelian group on admissible subcategories. First we need to define what "proper intersection" means in the category-theoretic context.

Let $A, B$ be two admissible subcategories such that $A = K_Y$ and $B = K_Z$ for two closed irreducible subsets $Y, Z \in \mathrm{Spc}(K)$. Then we claim that the intersection $A \cap B$ is equal to $K_{Y\cap Z}$. Indeed, recall that $K_Y = \{a \in K|\mathrm{supp}(a) \subset Y\}, K_Y = \{b \in K|\mathrm{supp}(b) \subset Z\}$ and $K_{Y\cap Z} = \{c \in K|\mathrm{supp}(c) \subset Y \cap Z\}$. We have that $A \cap B = K_Y \cap K_Z = \{c \in K|\mathrm{supp}(c) \subset Y \text{ and } \mathrm{supp}(c) \subset Z\} = \{c \in K|\mathrm{supp}(c) \subset Y \cap Z\} = K_{Y\cap Z}$. Now we make the following definition

**Definition 5.1.** *Let $A, B$ be two admissible subcategories. We say that* $A$ and $B$ intersect properly, *if $codim(supp(A \cap B)) = codim(supp(A)) + codim(supp(B))$.*

**Remark:** For a set $X \subset \mathrm{Spc}(K)$ we set $\mathrm{codim}(X) = \dim(\mathrm{Spc}(K)) - \dim(X)$.

Next, let $J = K_Y, L = K_Z$ be admissible subcategories. Assume that $J$ and $L$ intersect properly and denote by $W_j, j \in J$ the irreducible components of the intersection $Y \cap Z$. Then define

$$i(W_j) = \sum_k (-1)^k \mathrm{length}(\mathrm{Tor}_k^A(A/I_{Y;W_j}, A/I_{Z;W_j}))$$

where $A$ is the local ring of $W_j$ at its generic point (which exists by proposition 2.8) and $I_{Y;W_j}$ and $I_{Z;W_j}$ are the local ideals of $Y$ and $Z$ at this point. This is in complete analogy to the intersection multiplicity as defined in the classical case (cf. [**Har77**, Appendix A]). Then define

$$J \cdot L := \sum_j i(W_j) K_{W_j}$$

As before, it is not clear that the newly defined intersection multiplicities are finite and we can circumvent this problem and state a corresponding conjecture as we did in the previous section. However, we can easily check that this definition recovers the original definition of the intersection product, if we choose $K = D^{\mathrm{perf}}(X)$ for a smooth projective variety $X$:

**Theorem 5.2.** *Let $X$ be a smooth projective variety over an algebraically closed field $k$. Then the intersection product on the admissible subcategories of $D^{perf}(X)$ defined* *above agrees with the usual intersection product on irreducible subvarieties of $X$ via the bijection given in 2.9 and the isomorphism Spec*$(D^{perf}(X)) \cong X$.

Proof. First, we check that two admissible subcategories intersect properly if and only if their geometric counterparts do. This follows immediately from definition 5.1, the fact that $K_Y \cap K_Z = K_{Y \cap Z}$ (which we showed above) and the fact that two subvarieties $Y, Z$ intersect properly if $\mathrm{codim}(Y \cap Z) = \mathrm{codim}(Y) + \mathrm{codim}(\mathrm{supp}(Z))$.

Now take two admissible subcategories $A = K_Y, B = K_Z$ that intersect properly and look at their intersection product $\sum_j i(W_j)K_{W_j}$. Via the bijection 2.9, it is clear that the corresponding intersection product for $Y \cdot Z$ will be of the form $\sum_j i(W_j)'W_j$. Now all there is to do is check that $i(W_j)' = i(W_j)$ for all $j$, which is clear as Spec $(D^{\mathrm{perf}}(X)) \cong X$. On the other hand, take two subvarieties $V, W$ that intersect properly. We already know that $K_V$ and $K_W$ will intersect properly and as Spec $(D^{\mathrm{perf}}(X)) \cong X$, we will have $K_V \cdot K_W = \sum_j i(U_j)K_{U_j}$, where $U_j$ are the irreducible components of $V \cap W$. The equality of the intersection multiplicities is a direct consequence of the isomorphism Spec $(D^{\mathrm{perf}}(X)) \cong X$ and their definition. □

CHAPTER 4

# Additional theory

In this chapter, we present some results that are related to our previous results but are not used to prove them. They show how to transport more geometrical notions to certain adequate tensor-triangulated categories.

## 1. Geometric Categories and Hilbert functions

We begin with a definition that will help us define a generalization of the Hilbert polynomial for a triangulated category endowed with cohomology and a Serre functor.

**Definition 1.1.** *A* geometric category *is a pair* $((K, \otimes, \mathbb{I}), H)$*, where*

- $(K, \otimes, \mathbb{I})$ *is a finite-dimensional, k-linear tensor-triangulated category;*
- $H$ *is a cohomological functor* $K \longrightarrow$ *Mod(Spec(*$K$*), where Mod(Spec(*$K$*)) is the category of* $\mathcal{O}_K$*-modules on Spec(*$K$*);*
- $K$ *is a equipped with a unique Serre fuctor* $S$*. (The uniqueness could, for example, be achieved if* $K$ *is also Hom-finite).*

The previous definition enables us to get hold of a function that is a generalization of the Hilbert polynomial of an algebraic variety:

**Definition 1.2.** *Let* $((K, \otimes, \mathbb{I}), H)$ *be a geometric category. Then define its* almost-Hilbert function*, denoted by* $aHf_K$ *as follows: for* $m \in \mathbb{Z}$ *set* $\mathcal{O}(m) := H(S^m(\mathbb{I})[-mn]) \in$ *Mod(Spec(*$K$*)), where* $\mathbb{I}$ *is the unit object of* $K$ *and* $n$ *is the dimension of the topological space Spc(*$K$*). Then* $aHf_K(k) = \chi(\mathcal{O}(k))$*, where* $\chi$ *denotes the Euler characteristic.*

**Remark 1:** As $K$ is $k$-linear, the cohomology groups $H^n(\mathcal{O}(k))$ are $k$-vector spaces and thus we can compute Euler characteristics.

**Remark 2:** By definition, we have dim $(\mathrm{Spc}(K)) < \infty$, and therefore $\mathcal{O}(k)$ is well-defined. However, it may happen that $\mathrm{aHf}_K(k) = \chi(\mathcal{O}(k)) = \infty$.

**Example:** Let $X$ be smooth projective variety with very ample canonical bundle and look at $D^b(X) = D^{\mathrm{perf}}(X)$. Then $(D^b(X), H^0)$, where $H^0$ is the standard 0-th cohomology, is a geometric category: indeed, Balmer's reconstruction says that $X \cong \mathrm{Spec}(D^b(X))$ and thus $H$ has the right codomain. $D^b(X)$ is also equipped with a Serre functor, which is given by $\otimes\omega_X[n]$, where $n = \dim(X)$. Now, $\mathcal{O}(m) =$

$$H^0(\mathcal{O}_X[0]\otimes\omega_X^{\otimes m}[mn][-mn]) = H^0(\mathcal{O}_X[0]\otimes\omega_X^{\otimes m}) = H^0(\mathcal{O}_X[0])\otimes\omega_X^{\otimes m} = \mathcal{O}_X\otimes\omega_X^{\otimes m} \cong \mathcal{O}_X(m)$$

where $\mathcal{O}_X(m)$ is the $m$-th twisting sheaf of Serre and we used that cohomology commutes with tensoring with a locally free sheaf and that $\omega_X$ is very ample. Therefore, in this case, we see that the almost-Hilbert function of $D^b(X)$ coincides with the Hilbert polynomial of $X$ relative to the embedding given by $\omega_X$ (cf. [**Har77**, Exercise III.5.2]). Note that if we relax the condition on $\omega_X$ to be only ample, then we get a function $P'(x)$ such that the Hilbert polynomial $P(x)$ is given by $P'(dx)$ for some fixed $d \in \mathbb{Z}$.

With the help of almost-Hilbert functions we can extend various notions of geometric invariants to geometric categories. For example, we can make the following definition:

**Definition 1.3.** *Let $(K, H)$ be a geometric category such that $K$ is equipped with a Serre functor $S$. Then define the* arithmetic genus of $(K, H, S)$ *as* $p_a(K, H, S) = aHf_K(0)$.

**Example:** Let $K = D^b(X)$, where $X$ is a smooth projective variety with ample canonical sheaf. In the previous example, we have seen that $aHf_{D^b(X)}$ is a function $P'(x)$ such that the Hilbert polynomial $P_X(x)$ is given by $P'(dx)$ for some fixed $d \in \mathbb{Z}$. Now we see that $p_a(X) = P_X(0) = P'(0)$ and thus we see that we have recovered the arithmetic genus of $X$.

## 2. Geometric product categories

For two smooth projective varieties $X, Y$ we can look at the tensor-triangulated categories $D^b(X), D^b(Y)$ and ask if there is a way to generalize the "product category" $D^b(X \times_k Y)$ (note that this is *not* equal to the cartesian product $D^b(X) \times D^b(Y)$). It turns out that Balmer's reconstruction theorem makes the construction of such a category possible.

So far, we've only defined $D^{\mathrm{perf}}(X)$ for $X$ an algebraic variety. The definition easily generalizes to arbitrary ringed spaces:

**Definition 2.1.** *Let $(X, \mathcal{O}_X)$ be a ringed space. A* strictly perfect complex *on $X$ is a complex $\mathfrak{a}$ of sheaves of $\mathcal{O}_X$-modules such that $\mathfrak{a}$ is locally isomorphic to a complex of free $\mathcal{O}_X$-modules of finite rank. A* perfect complex *is a complex of sheaves of $\mathcal{O}_X$-modules that is locally quasi-isomorphic to a strictly perfect complex. Denote by $D^{perf}(X)$ the full subcategory of $D^b(Mod(X))$ consisting of the perfect complexes.*

**Remark:** By the remark after definition 1.2 of the previous chapter, this is a sensible generalization, as for $X$ and algebraic variety, both definitions of $D^{\mathrm{perf}}(X)$ coincide.

For arbitrary ringed spaces, we no longer have $D^{\mathrm{perf}}(X) = D^b(X)$ as we have not defined the latter for this situation. Thus we will need to do a bit of work to make sure that $D^{\mathrm{perf}}(X)$ is triangulated. Furthermore, we also want to give it a tensor structure. Both statements are given by the following proposition:

**Proposition 2.2.** *Let $(X, \mathcal{O}_X)$ be a ringed space. Then $D^{perf}(X)$ is a tensor-triangulated category.*

Proof. It is easy to see that the left-derived tensor product $\otimes^L$ with unit $\mathcal{O}_X$ induces a tensor-structure on $D^{\mathrm{perf}}(X)$. Next, we prove that $D^{\mathrm{perf}}(X)$ is triangulated. By definition, we have that $D^{\mathrm{perf}}(X) \subset D^b(\mathrm{Mod}(X))$, it is a full subcategory of the bounded derived category of $\mathcal{O}_X$-modules. The latter is triangulated as $\mathrm{Mod}(X)$ is abelian (cf. [**Har77**, Example III.1.0.4]) and thus we only have to check that $D^{\mathrm{perf}}(X)$ is stable under isomorphisms, shifts and taking cones (cf. [**Nee01**, Definition 1.5.1]). It is clear that a complex isomorphic to a perfect complex is perfect and that the shift of a perfect complex is still perfect. Thus, it remains to prove that the mapping cone of a morphism of perfect complexes is in $D^{\mathrm{perf}}(X)$. Thus, let $f : A^\bullet \longrightarrow B^\bullet$ be a morphism in $D^{\mathrm{perf}}(X)$, then the mapping cone $C(f)$ is given by the complex

$$C(f)^i := A^{i+1} \oplus B^i \text{ with differentials } d^i_{C(f)} := \begin{pmatrix} -d^{i+1} & 0 \\ f^{i+1} & d^i_B \end{pmatrix}$$

The direct sum of two perfect complexes is perfect as the direct sum of two locally free sheaves is locally free. This shows that $D^{\mathrm{perf}}(X)$ is stable under taking cones and therefore concludes the proof. □

Now we are ready to construct a tensor-triangulated category that generalizes the "product category" as mentioned at the beginning of this section.

**Definition 2.3.** *Let $K$, $L$ be tensor-triangulated categories that satisfy the conditions $(*)$ of chapter 3, section 4. The* geometric product category *of $K$ and $L$ is the category $P(K,L) := D^{perf}(Spec(K) \times_k Spec(L))$, where $Spec(K) \times_k Spec(L)$ is the fibred product of the ringed spaces $Spec(K)$ and $Spec(L)$ over $Spec(k)$.*

**Remark:** By the previous proposition, $P(K,L)$ is tensor-triangulated. Note that we do *not* claim that $\mathrm{Spec}(P(K,L)) \cong \mathrm{Spec}(K) \times_k \mathrm{Spec}\ (L)$ in general and it seems an interesting open problem to find out exactly when this equality holds. However, we have the following example, which shows that it *is* true for the special case where $K, L$ come from varieties. Thus, in this sense our definition generalizes the "product category" as mentioned before.

**Example:** Let $X, Y$ be smooth projective varieties. Then $D^b(X \times Y)$ is the geometric product category of $D^b(X)$ and $D^b(Y)$: indeed, $\mathrm{Spec}(D^b(X)) \cong X$, $\mathrm{Spec}(D^b(Y)) \cong Y$ and $D^{\mathrm{perf}}(\mathrm{Spec}(D^b(X)) \times \mathrm{Spec}(D^b(Y))) = D^{\mathrm{perf}}(X \times Y) = D^b(X \times Y)$.

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