Exercises day 1, Rational homotopy theory

The first exercise will use the construction of the *Hopf invariant* of a map $f\colon S^{2n-1} \to S^n$, for $n > 1$. It is defined as follows: build a CW complex X_f by attaching a $2n$-cell to S^n along the map f. Then the cohomology of X_f is easily checked to be the following:

$$H^*(X_f) \cong \begin{cases} \mathbb{Z} & \text{if } * = 0,\ n,\ 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Write a for a generator of $H^n(X_f)$ and b for a generator of $H^{2n}(X_f)$. With respect to the cup product on cohomology, there must then be a relation of the form

$$a^2 = k \cdot b$$

for some integer k. This k is called the Hopf invariant of f, often denoted $H(f)$. (As described here this integer is only well-defined up to sign, depending on the choice of the generator b. This will not matter for our purposes.) In the following problem you can use that the Hopf invariant defines a homomorphism of abelian groups

$$H\colon \pi_{2n-1}S^n \to \mathbb{Z}.$$

Exercise 1. The aim of this exercise is to prove that the Whitehead bracket $[\iota_{2n}, \iota_{2n}]$ of the fundamental class of S^{2n} with itself represents a nonzero element of the group $\pi_{4n-1}(S^{2n}) \otimes \mathbb{Q}$. Do this by showing that the Hopf invariant of $f := [\iota_{2n}, \iota_{2n}]$ is 2 (or -2, depending on your choices). (Hint: use the definition of the Whitehead product from the attaching map of the $4n$-cell of $S^{2n} \times S^{2n}$ to first construct a map $S^{2n} \times S^{2n} \to X_f$. Then examine the effect of this map on cohomology.)

Exercise 2. In this exercise we will explore how to construct rationalizations of spaces explicitly.

(a) Consider a sphere S^n, $n \geq 1$, and pick a self-map $p\colon S^n \to S^n$ of degree p. Prove that the mapping telescope (look up this notion up if you don't know it) of this self-map is a space with reduced homology concentrated in degree n, where it takes the value $\mathbb{Z}[\frac{1}{p}]$. Combine these constructions for all primes p to produce a rationalization of S^n.

(b) Try to use the construction of part (a) to explicitly construct a rationalization for every CW complex X (with cells starting in dimension 2, for simplicity).

(c) Consider an Eilenberg–MacLane space $K(A, n)$, with A an abelian group and $n \geq 2$. (Recall that $K(A, n)$ is a connected space with nth homotopy group A and all other homotopy groups 0.) Describe a rationalization of this space.

(d) Use your answer for (c) to construct a rationalization for a general simply-connected space. (Think about the Postnikov tower of X.)

Exercise 3. Let X be a simply-connected CW complex that is rational. Prove that if X is not contractible, then it has to consist of infinitely many cells.

Exercise 4. (For those who like categories.) Write i for the inclusion

$$\mathrm{Ho}(\mathcal{S}_{\mathbb{Q}}^{\geq 2}) \to \mathrm{Ho}(\mathcal{S}^{\geq 2})$$

of the rational homotopy category into the homotopy category of all simply-connected CW complexes. Prove that rationalization of spaces defines a functor

$$L_{\mathbb{Q}}\colon \mathrm{Ho}(\mathcal{S}^{\geq 2}) \to \mathrm{Ho}(\mathcal{S}_{\mathbb{Q}}^{\geq 2})$$

in the opposite direction that is left adjoint to i. Now prove that $L_{\mathbb{Q}}$ exhibits the rational homotopy category $\mathrm{Ho}(\mathcal{S}_{\mathbb{Q}}^{\geq 2})$ as the localization of $\mathrm{Ho}(\mathcal{S}^{\geq 2})$ at the rational equivalences, i.e., that the former is the category obtained from the latter by formally adding inverses for rational equivalences.