Exercises day 3, Rational homotopy theory

Exercise 1. Prove that the free Lie algebra on a generator x_n of degree n (equipped with the zero differential) is a Lie model for the sphere S^{n+1}.

Exercise 2. Find a Lie model for $SU(n)$.

In the following exercise you may want to use the following recipe,which essentially says that assigning Lie models to spaces preserves pushouts of a nice enough kind. Suppose X is a space, $f\colon S^n \to X$ a map, and Y the space obtained from X by attaching an $(n+1)$-cell along the map f. If L is a Lie model for X and $\varphi \in L_{n-1}$ is a cycle representing the homotopy class of f, then a Lie model for Y is given by adding a generator y of degree n to L, together with the relation $dy = \varphi$.

Exercise 3.

(a) Recall that the free Lie algebra on a generator x_1 of degree 1 is a Lie model for S^2. Find a cycle φ of degree 2 representing the Hopf map $\eta\colon S^3 \to S^2$.

(b) Use the result of (a) to prove that a Lie model for $\mathbb{C}P^2$ is given by the free Lie algebra on generators x_1 and x_3 with the relation $dx_3 = \frac{1}{2}[x_1, x_1]$. Does this Lie algebra give the correct rational homotopy groups of $\mathbb{C}P^2$, as computed yesterday?

Exercise 4. (A bit more challenging.) Generalize the result of the previous problem and show that a Lie model for $\mathbb{C}P^n$ is given by the free Lie algebra on generators $\xi_1, \xi_2, \ldots, \xi_n$ with ξ_k of degree $2k-1$ and differential given by

$$d\xi_k = \sum_{a+b=k} \frac{1}{2}[\xi_a, \xi_b].$$