Parallels between moduli of quiver representations and vector bundles

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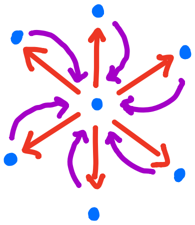

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§1 Overview

Moduli spaces are geometric solutions to classification problems

Given: a set $\mathcal{A}$ of objects we want to classify
an equiv. relation $\sim$ on $\mathcal{A}$

We want to equip $\mathcal{A}/\sim$ with a geom structure describing how objects vary continuously in families

History: Riemann coined the term "moduli" whilst classifying compact Riemann surfaces up to isomorphism.

Our key examples

1) Moduli of quiver representations

2) Moduli of vector bundles on a smooth projective curve
(Over $k = \mathbb{C}$: compact Riem. surface)

We'll see these moduli problems have many similarities:

- They are 1-dimensional (associated abelian categories have homological dimension 1)
- They have associated smooth moduli stacks and smooth moduli spaces of "stable objects"
- They have moduli spaces constructed as quotients of algebraic group actions via "Geometric Invariant Theory"

Over $k = \mathbb{C}$

- These moduli spaces are Kähler and are constructed symplectically via Hamiltonian reduction
- They have associated hyperkähler moduli spaces
- There are surprising results relating

counts of abs. indecomposable objects in these moduli problems over finite fields ⟷ dimensions of cohomology groups of their associated hyperkähler moduli spaces

due to

Crawley-Boevey & Van den Bergh
Hausel, Letellier & Rodriguez-Villegas } for quivers
Schiffmann for vector bundles

Furthermore, these moduli problems have associated Hall algebras which play an important role in computing enumerative invariants ("Donaldson-Thomas invariants") and in representation theory (Sadly we won't touch on these aspects)

§2 Moduli spaces of quiver representations

A quiver $Q = (V, A, s, t : A \to V)$ is a directed graph

vertices ↗ arrows ↑ ↖ source and target of each arrow

Def: A k-representation of $Q = (V, A, s, t : A \to V)$

is $W = \left(W_v, v \in V, \ \varphi_a : W_{s(a)} \longrightarrow W_{t(a)}, a \in A \right)$

(finite dim^{e}) k-vector spaces ↗ ↖ k-linear maps

The dimension vector of W is $\dim W = (\dim W_v)_{v \in V} \in \mathbb{N}^V$

Ex: Jordan Quiver

k-rep of Q is
$$\begin{tikzcd} W \arrow[loop above, "f"] \arrow[d, phantom, sloped, "\cong"] \\ {} \end{tikzcd}$$
— Choose basis

$$N \sim M \iff N \circlearrowright k^n \cong k^n \circlearrowleft M$$

Rmk: For k-reps W & W' of Q, we have

$\mathrm{Hom}_Q(W, W')$ morphisms $f : W \to W'$ (ie. $f_v : W_v \to W'_v$ k-linear $\forall v \in V$ fitting into comm squares $\forall a \in A$)

$\mathrm{Ext}^1_Q(W, W')$ classes of extensions of W by W'
s.e.s $0 \to W' \longrightarrow U \longrightarrow W \to 0$

There is an abelian category $\mathrm{Rep}(Q, k)$ of k-reps of Q.

Euler form: $\langle W, W' \rangle_Q := \dim \mathrm{Hom}_Q(W, W') - \dim \mathrm{Ext}^1_Q(W, W')$

Exercises: In the lecture notes do

Ex 2.3 (computing $\langle W, W' \rangle_Q$ in terms of $\dim W$ & $\dim W'$)

Ex 2.6 (showing $\mathrm{Rep}(Q, k)$ has homological dim $= 1$)

Quiver moduli: classify k-reps of Q of dim d / $\cong$

$$G := \prod_{v\in V} GL(d_v, k) \circlearrowright Rep := \bigoplus_{a\in A} Hom(k^{d_{s(a)}}, k^{d_{t(a)}})$$

Note: G & Rep are (affine) algebraic varieties over k
vanishing loci of polynomials

Constructing a quotient ⇝ "moduli space"

Algebraically: Geometric Invariant Theory (GIT)

Given an alg group action $G \times X \to X$

STEP 1: Associate to $G \circlearrowright X$, a ring $R(X)$ of "functions on X" with a G-action.

STEP 2: Take the ring of G-invariant functions

$$R(X)^G \subset R(X).$$

STEP 3: Define the quotient $X/\!/G$ as the variety corresponding to $R(X)^G$.

OUTCOME:

$$\begin{tikzcd} X \arrow[r, dashed] & X/\!/G \\ X^{ss} \arrow[u, phantom, "\cup"] \arrow[ur, "q"] & \end{tikzcd}$$

— GIT quotient

↖ not nec an orbit space!

"semistable" locus ←

(In general hard to describe)

constant on orbits

⚠ Some orbits identified
Some "unstable" orbits removed

Ex (Jordan quiver, $d=2$)

$G = GL_2 \curvearrowright X = \mathrm{Mat}_{2\times 2}$ $/k=\mathbb{C}$

$R(X) = \mathbb{C}[x_{11}, x_{12}, x_{21}, x_{22}] \supset R(X)^G = \mathbb{C}[\det, \mathrm{tr}]$

$q: X^{ss} = X \longrightarrow X/\!/G = \mathbb{C}^2 \quad , \quad M \mapsto (\det M, \mathrm{tr}\, M)$

↳ identifies some orbits (JNFs): $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$ and $\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$

Exercises: Do Exercises 2.14 & 2.15

Thm (Le Bruyn & Procesi) The invariant ring $\mathcal{O}(\mathrm{Rep})^G$ is generated by taking traces of oriented cycles.

Problem: For quivers without oriented cycles, $\mathcal{O}(\mathrm{Rep})^G = k$ and $\mathrm{Rep}/\!/G = *$

Solution (King): In Step 1, modify the G-action using a "stability parameter" $\theta \in \mathbb{Z}^V$

θ determines a character $\chi_\theta : G \longrightarrow \mathbb{G}_m$ ← multiplicative gp of k

$$(g_v)_{v\in V} \longmapsto \prod_{v \in V} (\det g_v)^{\theta_v}$$

which we use to lift the G-action to the trivial line bundle $\mathrm{Rep} \times \mathbb{A}^1$ by $g\cdot(r, z) = (g\cdot r, \chi_\theta(g) z)$

Let $\mathcal{L}_\theta$ denote this G-line bundle on Rep ("linearisation")

G acts on the ring $R = \bigoplus_{n \geq 0} H^0(\mathrm{Rep}, \mathcal{L}_\theta^{\otimes n})$ of sections of powers of $\mathcal{L}_\theta$

Let $R^G \subset R$ denote the invariant ring, then for each θ we obtain:

$$\begin{tikzcd} \text{Rep} \arrow[r, dashed] & \text{Rep}/\!/_\theta G := \text{Proj}(R^G) \\ \text{Rep}^{\theta\text{-ss}} \arrow[u, phantom, "\cup"] \arrow[ur] & \end{tikzcd}$$

θ-ss locus

GIT quotient w.r.t. θ

Rmk: i) For $\theta = 0$, $\text{Rep}/\!/_\theta G = \text{Rep}/\!/G := \text{Spec}(\mathcal{O}(\text{Rep})^G)$

ii) The 0^{th}-graded piece of R^G is $\mathcal{O}(\text{Rep})^G$ & so

$$\text{Rep}/\!/_\theta G \xrightarrow[\text{morphism}]{\text{proj.}} \text{Rep}/\!/G \leftarrow \text{affine variety}$$

iii) As $\Delta = \{(t\,\text{Id}_{d_v})_{v\in V} : t \in \mathbb{G}_m\} \curvearrowright \text{Rep}$ trivially, we need $\chi_\theta(\Delta) = 1$ for invariant sections of $\mathcal{L}_\theta^{\otimes n}$ to exist for $n > 0$

$$\Updownarrow$$

Assume $\boxed{\sum_{v\in V} \theta_v d_v = 0}$ (Otherwise the semistable set is empty)

Moduli theoretic description of stability

<u>Def</u>: A d-dim^e k-rep W of Q is

$\sum_{v\in V} \theta_v \dim W_v$ //

θ-<u>semistable</u> if $\sum_{v\in V} \theta_v \dim W'_v \geq 0$ for all proper subreps

(resp. θ-<u>stable</u>) (resp. > 0) $0 \neq W' \subsetneq W$

We say θ is <u>generic</u> w.r.t d if θ-semistability ⟺ θ-stability for d-dim^e reps of Q

(Do Exercise 2.20 & Exercise 2.23)

(i.e. $\sum_{v\in V} \theta_v d'_v \neq 0 \ \forall\, d' \neq d$ with $0 \leq d'_v \leq d_v$)

[King] GIT θ-semistability ⟺ moduli theor. θ-semistability

$\mathcal{M}_d^{\theta\text{-ss}}(Q) := \text{Rep}/\!/_\theta G$ moduli space of $\underset{\text{S-equiv classes of}}{\wedge}$ θ-ss d-dim^e k-reps of Q

Symplectic quotients ($k=\mathbb{C}$): Hamiltonian reduction

$U := \prod_{v\in V} U(d_v) < G = \prod_{v\in V} GL(d_v,\mathbb{C}) \curvearrowright Rep = \bigoplus_{a\in A} Hom(\mathbb{C}^{d_{s(a)}}, \mathbb{C}^{d_{t(a)}}) = \mathbb{C}^n$

$\uparrow$ max^{l} compact

Hermitian form $H(M,N) = \sum_{a\in A} Tr(M_a N_a^{\dagger}) \rightsquigarrow \omega := -Im\, H$ symplectic form

(In fact, kähler)

non-deg skew-sym bilinear form

The U-action on (Rep, ω) is Hamiltonian with moment map $\mu_{\mathbb{R}} : Rep \to \mathfrak{u}^* = (Lie\ U)^*$

smooth, U-equivariant, lifts the infinitesimal action

(Do Exercise 2.28: explicit description of $\mu_{\mathbb{R}}$)

Hamiltonian reduction: $\mu_{\mathbb{R}}^{-1}(0)/U$ + induced sympl. form (if smooth)

[Marsden-Weinstein, Meyer]

Stab par. $\theta \in \mathbb{Z}^V \rightsquigarrow \theta \in \mathfrak{u}^* \rightsquigarrow \mu_{\mathbb{R}}^{-1}(\theta)/U$

Kempf-Ness Thm

symplectic quotient $\mu_{\mathbb{R}}^{-1}(\theta)/U \simeq Rep /\!/_{\theta} G$ algebraic quotient

Thm (H) ($k=\mathbb{C}$) Three natural stratifications of Rep agree:

1) GIT: by GIT instability (using torus weights)
2) Symplectic: Morse stratification assoc. to $\|\mu\|^2$
3) Moduli: using filtrations by subrepresentations.

Ex:  $d=(1,1)$

$$\begin{tikzcd} G = \mathbb{C}^\times \times \mathbb{C}^\times \arrow[d, phantom, "\vee"] \curvearrowright \mathrm{Rep} = \bigoplus_{a\in A} \mathrm{Hom}(\mathbb{C},\mathbb{C}) = \mathbb{C}^n \ni 0 \\ U = S^1 \times S^1 \end{tikzcd}$$

lies in the closure of each orbit

Note: diagonal torus acts trivially, so can consider

$\Delta < G$

$$\begin{tikzcd} G = \mathbb{C}^\times \arrow[d, phantom, "\vee"] & \curvearrowright \mathrm{Rep} = \mathbb{C}^n \arrow[d, "\mu_{\mathbb{R}}"] & \ni (z_1,\dots,z_n) \arrow[d, mapsto] \\ U = S^1 & \mathfrak{u}^* \cong \mathbb{R} & \ni \sum_{j=1}^n |z_j|^2 \end{tikzcd}$$

For $\theta = (-1,1) \in \mathbb{Z}^V$

Kempf-Ness homeo

$$\mu_{\mathbb{R}}^{-1}(\theta)/S^1 = S^{2n-1}/S^1 \simeq \mathbb{P}^{n-1}_{\mathbb{C}} \simeq (\mathbb{C}^n - \{0\})/\mathbb{C}^\times = \mathrm{Rep}^{\theta\text{-ss}}/\mathbb{C}^\times = \mathrm{Rep}/\!/_{\theta}\, \mathbb{C}^\times$$

Further Examples of quiver moduli spaces

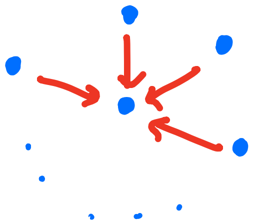
moduli of r-planes in k^n

- Grassmannians $Gr(r,n)$ are quiver moduli spaces

$d=(1,r)$

$$\begin{tikzcd} GL_r \curvearrowright \mathrm{Rep} = \mathrm{Mat}_{r\times n} \arrow[r, dashed] & \mathrm{Rep}/\!/_{\theta} GL_r = Gr(r,n) \\ \mathrm{Rep}^{\theta\text{-ss}} = \mathrm{Mat}^{rk=r}_{r\times n} \arrow[u, phantom, "\cup"] \arrow[ur, two heads] & \end{tikzcd}$$

- Configuration spaces of n points in $\mathbb{P}^1_k$



$(k=\mathbb{C})$ Symplectic interpretation: moduli spaces of polygons

- Moduli spaces of instantons ⇝ Hyperkähler quiver moduli spaces (lecture 2)

§3 Moduli spaces of vector bundles on curves

Let X be a smooth projective algebraic curve $/k$

no singularities ↵

Jacobi criterion: for all $p \in X$

$\operatorname{rk}\left(\frac{\partial f_i}{\partial x_j}(p)\right) = n-1$

↓

$X = V(f_1,..f_m) \subset \mathbb{P}^n$

vanishing locus of homog. polys.

↳ 1-dim^{e} variety

Assume: $k = \overline{k}$

(for simplicity)

(Over $k=\mathbb{C}$: X = compact Riemann surface)

Genus of X: $g = \dim H^0(X, \omega_X)$ (= topological genus)

canonical line bundle → ω_X

$$\omega_X = \Omega^1_X$$

cotangent bundle: sheaf of 1-forms

Def: An (algebraic) vector bundle over X of rank n is a morphism $\pi: E \to X$ of alg varieties s.t $\exists$ (Zariski) open cover $X = \bigcup_{i \in I} U_i$ and $\forall\, i \in I$ $\exists$ local trivialisations

$$\psi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{A}^n_k \quad \text{s.t} \quad pr_{U_i} \circ \psi_i = \pi|_{\pi^{-1}(U_i)}$$

and $\forall\, i,j$

$$\psi_j \circ \psi_i^{-1}|_{U_i \cap U_j} : U_i \cap U_j \times \mathbb{A}^n_k \longrightarrow U_i \cap U_j \times \mathbb{A}^n_k$$
$$(x, v) \longmapsto (x, g_{ij}(x)v)$$

with transition functions $g_{ij} : U_i \cap U_j \longrightarrow GL_n$.

Ex:
- $X \times \mathbb{A}^n_k \xrightarrow{pr_1} X$ is the trivial rk n vector bundle
- The (co)tangent bundle on X is a line bundle (rank 1 vector bundle)

Rmk Constructions in linear algebra carry over to vector bundles$/X$:

$$\left.\begin{array}{l} E\to X \text{ rk } n \\ F\to X \text{ rk } m \end{array}\right\} \rightsquigarrow \begin{array}{lll} \text{direct sums} & E\oplus F\to X & \text{rk } n+m \\ \text{tensor products} & E\otimes F\to X & \text{rk } nm \\ \text{duals} & E^\vee & \text{rk } n \\ \text{exterior powers:} & \det(E)=\Lambda^n E & \text{rk } 1 \end{array}$$

$\rightsquigarrow$ notions of subbundles & quotient bundles & vector bundle homomorphisms

Rmk: We have equivalences of categories

$$\begin{tikzcd} \mathrm{Vect}^{hol}(X) \arrow[r, phantom, "\overset{k=\mathbb{C}}{\simeq}"] & \mathrm{Vect}(X) \arrow[r, phantom, "\simeq"] & \mathrm{Coh}^{l.f.}(X) \arrow[d, hook] \\ & & \mathrm{Coh}(X) \end{tikzcd}$$

GAGA

holomorphic vector bundles on the RS X

$X\times\mathbb{A}^1$ trivial line bundle $\longleftrightarrow$ $\mathcal{O}_X$ structure sheaf

locally free coherent sheaves on X

abelian cat of hom. dim 1

$$\left[\mathrm{Ext}^i(\mathcal{E},\mathcal{F}) = H^i(X,\mathcal{H}om(\mathcal{E},\mathcal{F}))=0 \quad \forall\ i>\dim(X)=1.\right] \hookleftarrow$$

Every coherent sheaf on X (smooth proj curve) is an extension of a locally free sheaf (vector bundle) by a "torsion sheaf" $\hookleftarrow$ ie a finite sum of "skyscraper sheaves" at $x\in X$ $\hookleftarrow$

Notation: k_x sections over $U\subset X$ $\quad k_x(U)=\begin{cases} 0 & x\notin U \\ k & x\in U\end{cases}$

There is a s.e.s of coherent sheaves

$$0\to\mathcal{O}_X(-x)\longrightarrow\mathcal{O}_X\longrightarrow k_x\to 0$$

line bundle

skyscraper sheaf at x

sections over $U\subset X$: $\mathcal{O}_X(-x)(U)=\{s\in\mathcal{O}_X(U) : s(x)=0 \text{ if } x\in U\}$

Relationship between divisors and line bundles on X

$$\begin{tikzcd} \mathrm{Div}(X) \arrow[d, "\deg"] & = \{ \sum_{p\in X} m_p\cdot p \arrow[d, mapsto] & : m_p \in \mathbb{Z} \text{ only finitely many non-zero} \} \\ \mathbb{Z} & \ni \sum_{p\in X} m_p & \end{tikzcd}$$

gp under addition

For $D = \sum_{p\in X} m_p\cdot p$, $\exists$ line bundle $\mathcal{O}_X(D)$ & s.e.s.

$$0 \to \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow k_D = \bigoplus_{p\in X} k_p^{\oplus m_p} \longrightarrow 0$$

dual of $\mathcal{O}_X(D)$

↳ sections vanish at p to order m_p

⇝ linear equiv: $D \sim E \;:\Longleftrightarrow\; \mathcal{O}_X(D) \cong \mathcal{O}_X(E)$

(Over $k = \mathbb{C}$: D & E differ by the divisor of a mero function)

$\mathrm{Jac}(X) := \mathrm{Div}^0(X)/\sim$ ← linear equiv classes of deg 0 divisors

↑ group of degree 0 line bdles on $X/\cong$

Jacobian variety, smooth proj comm gp / k

↳ $\dim \mathrm{Jac}(X) = g$ genus of X

Construction of Jac(X) over $k = \mathbb{C}$

We can integrate holo 1-forms ω along paths γ in X

Fix $x \in X$: for any $p \in X$ and $\gamma : x \rightsquigarrow p$, $\int_\gamma \omega \in \mathbb{C}$

integral depends on choice of path

Abel-Jacobi map

$$\begin{aligned} \alpha : X &\longrightarrow \mathrm{Jac}(X) := H^0(X, \Omega^1_X)^* / H_1(X, \mathbb{Z}) \cong \mathbb{C}^g/\mathbb{Z}^{2g} \\ p &\longmapsto [\omega \mapsto \textstyle\int_x^p \omega] \end{aligned}$$

cx torus of dim g

extends to divisors:

$$\begin{aligned} \mathrm{Div}^d(X) &\longrightarrow \mathrm{Jac}(X) \\ \sum_{p\in X} m_p\cdot p &\longmapsto \sum_{p\in X} m_p\cdot\alpha(p) \end{aligned}$$

inducing iso $\mathrm{Div}^d(X)/\sim \;\cong \mathrm{Jac}(X)$

Moduli of vector bundles: classify vector bundles on $X/\cong$

Fix invariants : rank n, degree d

Def The degree of $E \to X$ is $\deg(E) = \deg(\overset{\text{line bundle}}{\det E})$

Rmk: The Riemann-Roch Thm expresses the Euler characteristic of E in terms of the rank n & degree d of E:

$$\chi(E) := \dim H^0(X,E) - \dim H^1(X,E) = nd + n(1-g)$$

Serre duality: $H^1(X,E) \cong H^0(X, E^\vee \otimes \omega_C)^*$

Euler form: $\langle E,F\rangle := \chi(E^\vee \otimes F) = n_E d_F - n_F d_E + n_E n_F (1-g)$

Exercise: prove the equality (ie $\deg(E \otimes F) = d_E n_F + d_F n_E$)

Ex: The Jacobian variety $\mathrm{Jac}(X)$ is a moduli space for rank 1 degree 0 line bundles on $X/\cong$

For $n>1$
- the family of rk n deg d bundles is "unbounded"
- to obtain a moduli space: restrict to "semistable" vector bundles

Stability for vector bundles

Def: The slope of a vector bundle $E \to X$ is $\mu(E) := \frac{\deg E}{\operatorname{rk} E}$

E is semistable (resp. stable) if $\forall\ 0 \neq E' \subsetneq E$ subbdle: $\mu(E') \leq \mu(E)$ (resp. $<$)

Exercises: Do Exercises 3.3 & 3.4