**Moduli of vector bundles** : Classify vector bundles on $X$ $/\cong$

smooth projective curve

Fix invariants : rank $n$, degree $d$

**Def** The degree of $E \to X$ is $\deg(E) = \deg(\det E)$

**Riemann-Roch Thm:**

$$\chi(E) := \dim H^0(X,E) - \dim H^1(X,E) = d_E + n_E(1-g)$$

**Serre duality:** $H^1(X,E) \cong H^0(X, E^\vee \otimes \omega_C)^*$

**Euler form:** $\langle E,F \rangle := \chi(E^\vee \otimes F) = n_E d_F - n_F d_E + n_E n_F (1-g)$

**Exercise**: prove the equality ↗ (ie $\deg(E \otimes F) = d_E n_F + d_F n_E$)

**Ex**: The Jacobian variety $\mathrm{Jac}(X) \cong \mathrm{Div}^0(X)/\sim \; \overset{(k=\mathbb{C})}{\cong} \mathbb{C}^g/\Lambda$

is a moduli space for degree 0 line bundles on $X/\cong$

$S$ variety — family of line bundles on $X$ param by $S$ (up to iso) $\longleftrightarrow$ $S \longrightarrow \mathrm{Jac}(X)$

For $n > 1$
- the family of rk $n$ deg $d$ bundles is unbounded
- to obtain a moduli space: restrict to semistable vector bundles

## Stability for vector bundles

**Def** : The slope of a vector bundle $E \to X$ is $\mu(E) := \dfrac{\deg E}{\mathrm{rk}\, E}$

$E$ is semistable (resp. stable) if $\forall\ 0 \neq E' \subsetneq E$ subbdle : $\mu(E') \leq \mu(E)$ (resp. $<$)

**Exercises** : Do Exercises 3.3 & 3.4

## Algebraic Construction of vector bundle moduli spaces via GIT

**Boundedness Lemma**: For a semistable vector bundle $F$ of rank $n$ and degree $d > n(2g-1)$, we have

1) $H^1(X, F) = 0$
2) $F$ is generated by its global sections.
   ↳ ie $ev: H^0(X,F) \otimes \mathcal{O}_X \longrightarrow F$ is surjective

Pf Write $\mathcal{F}$ for the corr. locally free sheaf.

1) By Serre duality

$$H^1(X,\mathcal{F})^* \cong H^0(X, \mathcal{F}^\vee \otimes \omega_X) \cong \mathrm{Hom}(\mathcal{F}, \omega_X).$$

If $H^1(X,\mathcal{F}) \neq 0$, then $\exists$ a non-zero homo $\phi: \mathcal{F} \longrightarrow \omega_X$

Let $\underset{\nearrow \atop \mathrm{rk}=n-1}{E} \subset F$ be the vector subbundle generated by $\ker(\phi)$

$$\deg E \geqslant \deg \ker(\phi) \geqslant \deg \mathcal{F} - \deg \omega_X = d - (2g-2)$$

By semistability of $F$

$$\frac{d-(2g-2)}{n-1} \leqslant \mu(E) \leqslant \mu(F) = \frac{d}{n} \Rightarrow d \leqslant n(2g-2) \text{ ↯}$$

2) It suffices to show $\forall x \in X$ the map on stalks

$$H^0(X,\mathcal{F}) \otimes \mathcal{O}_{X,x} \to \mathcal{F}_x \quad \text{is surjective.}$$

This follows by considering the l.e.s in $H^*$ associated to

$$0 \to \underset{\nearrow \atop \text{semistable of rank } n \text{ degree } d-n}{\mathcal{F}(-x)} = \mathcal{F} \otimes \mathcal{O}_X(-x) \longrightarrow \mathcal{F} \longrightarrow F_x = \mathcal{F} \otimes k_x \longrightarrow 0$$

and applying 1) to $\mathcal{F}(-x)$. ☒

**Cor** $\mathcal{F}$ is a quotient of a free sheaf of rank $\chi(\mathcal{F})$

**Pf:** $ev : H^0(X,\mathcal{F}) \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{F}$ is surjective

and $\dim H^0(X,\mathcal{F}) = \chi(\mathcal{F})$ as $H^1(X,\mathcal{F}) = 0$. ∎

**GIT quotient construction:** fix $n$ and $d > n(2g-1)$

Let $N = d + n(1-g)$ $= \chi(\mathcal{F})$ for $\mathcal{F}$ of rank $n$, degree $d$

**Def/Thm** (Grothendieck)

There is a projective **Quot scheme** $Q = \mathrm{Quot}_X^{n,d}(\mathcal{O}_X^{\oplus N})$ parametrising rank $n$ degree $d$ quotients of $\mathcal{O}_X^{\oplus N}$ (up to equiv.)

**Not$^n$:** $\underset{\substack{\text{open}\ \cap \\ Q}}{Q^{\mu\text{-ss}}}$ $:= \left\{ q : \mathcal{O}_X^{\oplus N} \twoheadrightarrow \mathcal{E} : \begin{array}{l} \mathcal{E} \text{ ss loc. free sheaf} \\ H^0(q) \text{ is an iso} \end{array} \right\}$

Given a rk $n$, deg $d$ semistable vector bundle $\mathcal{F}$

$$\left.\begin{array}{l} ev : H^0(X,\mathcal{F}) \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{F} \\ \text{and a } \underline{\text{choice}} \text{ of iso } \underset{(*)}{H^0(X,\mathcal{F}) \cong k^N} \end{array}\right\} \rightsquigarrow \underset{\substack{\cap \\ Q^{\mu\text{-ss}}}}{q_{\mathcal{F}}} : \mathcal{O}_X^{\oplus N} \twoheadrightarrow \mathcal{F}$$

$GL_N \cong \mathrm{Aut}(\mathcal{O}_X^{\oplus N}) \curvearrowright Q$

$\cup$

$\mathbb{G}_m$ (diagonal) acts trivially

**Goal:** construct a quotient of the $SL_N$-action on $Q^{\mu\text{-ss}}$

(or equivalently, a $PGL_N$-quotient) via Geometric Invariant Theory

**Thm** (Seshadri; Newstead, Le Potier, Simpson..)

For the $SL_N$-action on $Q$ appropriately linearised

i) $Q^{ss} = Q^{\mu\text{-}ss}$ ie. GIT-ss $=$ $\mu$-ss

ii) $Q/\!/SL_N =: \mathcal{M}^{ss}_X(n,d)$ moduli space of (S-equiv classes of) semistable rank n degree d vector bundles$/X$

$\uparrow$ proj. var.

iii) If $(n,d)=1$, then semistability = stability and

$\mathcal{M}^{ss}_X(n,d)$ is smooth. (as $Q^{\mu\text{-}ss}$ is smooth)

$$\dim \mathcal{M}^{ss}_X(n,d) = n^2(g-1)+1 = -\chi(\mathcal{E},\mathcal{E})+1$$

$$\left(\text{Since } T_{[\mathcal{E}]}\mathcal{M}^{ss}_X(n,d) \cong \mathrm{Ext}^1(\mathcal{E},\mathcal{E})\right)$$

**Symplectic Construction**: via Gauge theory $(k=\mathbb{C})$
[Atiyah & Bott]

Moduli of holo vector bundles on a compact RS $X$

$\updownarrow$ $\dim_{\mathbb{C}}(X)=1$

cx vector bundle + holo structure ( = Dolbeault operator $\bar\partial$ )

Fix a cx vector bundle $E \to X$ & Hermitian metric $h$

$$\mathcal{C}(E) := \{\text{holo str. on } E\} \cong \{h\text{-unitary connections on } E\} =: \mathcal{A}(E,h)$$

$$\nabla^{0,1} \longleftarrow\!\shortmid\ \nabla : \Omega^0(X,E) \to \Omega^1(X,E)$$

$\circlearrowleft$ $\mathcal{G}_{\mathbb{C}} = \mathrm{Aut}(E)$ $\qquad$ $\bar\partial \longmapsto \nabla_{h,\bar\partial}$ Chern connection $\qquad$ $\circlearrowright$ $\mathcal{G} = \mathrm{Aut}(E,h)$

$\mathcal{A} = \mathcal{A}(E,h)$ is an $\infty$-dim$^{l}$ affine space with a symplectic form:

$$\omega_{\mathbb{R}} : T_\nabla \mathcal{A} \times T_\nabla \mathcal{A} \longrightarrow \mathbb{R}$$

$$(\beta , \gamma) \longmapsto \int_X \underbrace{\mathrm{Tr}(\beta \wedge \gamma)}_{\in\, \Omega^2(X)}$$

where $T_\nabla \mathcal{A} \cong \Omega^1(X, \mathrm{End}(E,h))$

The curvature is a moment map for $\mathcal{G} \curvearrowright (\mathcal{A}, \omega_{\mathbb{R}})$

$$\mu_{\mathbb{R}} : \mathcal{A} \longrightarrow \mathrm{Lie}\, \mathcal{G}^* = \Omega^0(X, \mathrm{End}(E,h))^*$$

$$\nabla \longmapsto -F_\nabla \in \overset{\mathrel{\text{\rotatebox{90}{$\cong$}}}}{\Omega^2(X, \mathrm{End}(E,h))}$$

**Def** A connection $\nabla$ on $E$ is Hermitian-Einstein if $\exists$ Hermitian metric $h$ s.t $\nabla$ is $h$-unitary & proj. flat

$\hookrightarrow \star F_\nabla = -ic\, \mathrm{Id}_E$

Hodge star   constant

Do Ex 3.11 : Show $c = \mu(E)$

Hint: $\deg(E) = \int_X \frac{i}{2\pi} \mathrm{Tr}(F_\nabla)$

Hitchin-Kobayashi correspondence [Uhlenbeck-Yau, Donaldson]

A holo bundle $(E, \bar\partial)$ is semistable $\iff$ $E$ admits a HE conn. Moreover, this connection is unique up to unitary gauge transf. $\mathcal{G}$

Hamiltonian reduction of $\mathcal{G} \curvearrowright (\mathcal{A}, \omega_{\mathbb{R}})$:

$$\mathcal{M}_X^{\text{proj-flat}} = \mu_{\mathbb{R}}^{-1}\left(i \tfrac{d}{n} \mathrm{Id}\right) / \mathcal{G}$$

$\exists$ symplectic structure (in fact kähler via $\mathcal{A} \cong \mathcal{C}$) on smooth locus

cx v.sp

m. space of unitary (twisted) reps of $\pi_1(X)$

Hitchin-Kobayashi corr. $\rightarrow$ $\wr\wr$ $\leftarrow$ Narasimhan-Seshadri Thm

A holo vector bundle is stable $\iff$ it comes from an irreducible unitary (twisted) rep of $\pi_1(X)$

$$\mathcal{M}_X^{ss}(n,d) = \mathcal{C}^{ss} /\!/ \mathcal{G}_{\mathbb{C}}$$

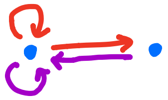
cx algebraic structure

# §4 Hyperkähler analogues of these moduli spaces

## Hyperkähler quiver moduli spaces $(k=\mathbb{C})$

Quiver $Q=(V,A,s,t:A\to V)$

Doubled quiver: for each $a\in A$ add opposite arrow

$Q$  $\rightsquigarrow$ $\bar{Q}=(V,\bar{A},s,t)$ 

$$\underset{\substack{\| \\ \mathbb{C}^n}}{\mathrm{Rep}_Q} = \bigoplus_{a\in A}\mathrm{Hom}(\mathbb{C}^{d_{s(a)}},\mathbb{C}^{d_{t(a)}}) \rightsquigarrow \mathrm{Rep}_{\bar{Q}} = \mathrm{Rep}_Q\times\mathrm{Rep}_Q^{*} = \underset{\substack{\cong \\ \mathbb{H}^n}}{T^{*}\mathrm{Rep}_Q}$$

$$\circlearrowleft \qquad \circlearrowleft \qquad \circlearrowright$$

$$\prod_{v\in V} U(d_v) = U < G = \prod_{v\in V} GL(d_v,\mathbb{C})$$

## Quaternions

$$\mathbb{H} = \mathbb{R}\oplus\mathbb{R}i\oplus\mathbb{R}j\oplus\mathbb{R}k \simeq \mathbb{C}\oplus\mathbb{C}j$$

$\mathbb{H}^n$ Hyperkähler : 3 complex structures $i,j,k$ and Riem. metric $g$ + 3 Kähler structures $\omega_i(\ ,\ ) = g(i-,-)$

Quaternionic inner product

$$q:\mathbb{H}^n\times\mathbb{H}^n\longrightarrow\mathbb{H} \qquad (z,w)\mapsto\sum_{l=1}^{n} z_l \overset{\dagger}{w_l}$$

quaternionic conjugate

$$q = g - i\omega_i - j\omega_j - k\omega_k \qquad \omega_{\mathbb{R}} = \omega_i \quad \text{real sympl}$$

$$= g - i\omega_{\mathbb{R}} - \omega_{\mathbb{C}}j \qquad \omega_{\mathbb{C}} = \omega_j + i\omega_k \quad \text{cx sympl}$$

## Hyperkähler structure on $\mathrm{Rep}_{\bar{Q}}$:

$$q(x,y) = \underbrace{\left(\sum_{a\in\bar{A}}\mathrm{Tr}(x_a y_a^{\dagger})\right)}_{g(x,y)-i\omega_{\mathbb{R}}(x,y)} - \underbrace{\left(\sum_{a\in A}\mathrm{Tr}(x_a y_{a^*} - x_{a^*}y_a)\right)}_{\omega_{\mathbb{C}}(x,y)} j$$

$x,y\in\mathrm{Rep}_{\bar{Q}}$

The U-action on $\mathrm{Rep}_{\bar{Q}} = \mathbb{H}^n$ is hyperhamiltonion
⇝ 3 moment maps

## Moment maps

$$\underset{\mu_i}{\overset{}{\mu_{\mathbb{R}}}} : \mathrm{Rep}_{\bar{Q}} \xrightarrow[\substack{\text{smooth} \\ U\text{-equivariant}}]{} \mathfrak{u}^* \qquad \underset{\mu_j + i\mu_k}{\overset{}{\mu_{\mathbb{C}}}} : \mathrm{Rep}_{\bar{Q}} \xrightarrow[\substack{\text{algebraic} \\ G\text{-equivariant}}]{} \mathfrak{g}^* = (\mathfrak{u} \oplus i\mathfrak{u})^*$$

$\mu_{\mathbb{R}} = \mu_i$, $\quad \mu_{\mathbb{C}} = \mu_j + i\mu_k$

If we identify $\mathfrak{u}^* \cong \mathfrak{u}$ and $\mathfrak{g}^* \cong \mathfrak{g}$ (via Killing form):

$$\mu_{\mathbb{R}}(X) = \frac{i}{2} \sum_{a \in \bar{A}} [X_a, X_a^+] \qquad \mu_{\mathbb{C}}(X) = \sum_{a \in A} [X_a, X_{a^*}] \quad (\text{ex})$$

**Hyperkähler reduction**: Choose $\theta \in \mathbb{Z}^V$, $\eta \in \mathfrak{g}^*$ coadjoint fixed

$$\mu_{\mathbb{R}}^{-1}(\theta) \cap \mu_{\mathbb{C}}^{-1}(\eta) \Big/ U \ \underset{\substack{\uparrow \\ \text{Kempf} \\ \text{Ness}}}{\simeq} \ \mu_{\mathbb{C}}^{-1}(\eta) /\!\!/_{\theta} \, G \quad \leftarrow \text{GIT quotient}$$

+ induced hyperkähler structure (if smooth)

**Rmk**: i) The RHS makes sense over any field $k$:

$\mathrm{Rep}_{\bar{Q}} = T^* \mathrm{Rep}_Q$ is alg. symplectic (+ Liouville form)

$\mu$ is an algebraic moment map

$\mu^{-1}(\eta) /\!\!/_{\theta} \, G$ is an algebraic symplectic reduction

ii) Special case: $\eta = (\eta_v)_{v \in V} \in \mathbb{C}^V$

$\mu_{\mathbb{C}}^{-1}(\eta) = \mathrm{Rep}_{\bar{Q}, R_\eta} \subset \mathrm{Rep}_{\bar{Q}}$ subvariety of $\bar{Q}$-reps satisfying rel^ns:

$$R_\eta = \Big\{ \sum_{a : s(a) = v} X_a X_{a^*} - \sum_{a : t(a) = v} X_{a^*} X_a = \eta_v \, \mathrm{Id}_{d_v}, \ v \in V \Big\}$$

iii) Proudfoot refers to the hyperkähler reduction at $\eta = 0$ as the hyperkähler analogue of $\mathcal{M}_d^{\theta\text{-ss}}(Q)$ because

$$\underset{\text{dense open}}{T^*\mathcal{M}_d^{\theta\text{-ss}}(Q) \subset} \mathcal{M}_d^{\theta\text{-ss}}(\bar{Q}, R_0) = \underset{\substack{\uparrow \\ \text{HK m.space}}}{\mu_{\mathbb{C}}^{-1}(0)} /\!/_{\theta} G$$

for $\theta$ generic.

**Higgs bundles**: hyperkähler analogue of vector bundles

Goal: HK moduli space $\underset{\text{dense open}}{\supset} T^*\mathcal{M}_X^{ss}(n,d)$

for $\mathcal{M}_X^{ss}(n,d)$ = moduli space of semistable rk $n$, degree $d$ vector bundles on $X$ (sm. proj. curve)

$$T_{[\mathcal{E}]}\mathcal{M}_X^{ss}(n,d) \cong \operatorname{Ext}^1(\mathcal{E},\mathcal{E}) \cong H^1(X, \operatorname{End}(\mathcal{E}))$$

Thus $T^*_{[\mathcal{E}]}\mathcal{M}_X^{ss}(n,d) \cong H^1(X, \operatorname{End}(\mathcal{E}))^* \underset{\substack{\uparrow \\ \text{Serre duality}}}{\cong} H^0(X, \operatorname{End}(\mathcal{E}) \otimes \omega_X)$

Def: A Higgs bundle on $X$ is a pair $(E, \Phi)$ consisting of a vector bundle $E \to X$ and a homomorphism $\Phi : E \to E \otimes \omega_X$ "Higgs field"

$(E, \Phi)$ is semistable (stable) if $\mu(E') \leq \mu(E)$ $(<)$ $\forall\, 0 \neq E' \subsetneq E$ Higgs subbdle ie $E'$ is $\Phi$-inv.

Rmk If $E$ is a semistable vector bundle, then $(E, \Phi)$ for any Higgs field $\Phi$ is a semistable Higgs bdle. Conversely there are unstable vector bundles which admit a Higgs field s.t the assoc Higgs bdle is stable (Exercise 4.10)

# Construction of Higgs moduli space

**Algebraic**: via GIT (see Simpson's papers)

**Gauge theoretic**: Hyperkahler reduction of $\mathcal{G} \curvearrowright T^*\mathcal{A}$

[Hitchin] $\mathcal{G} \curvearrowright \mathcal{A}$ unitary gauge gp ↗ $\mathcal{A}$ = space of unitary connections

Recall

$$\mathcal{G}_\mathbb{C} \curvearrowright \mathcal{C} = \{\bar\partial \text{ holo str}\} \underset{[A\text{-}B]}{\cong} \{\nabla \text{ unitary conn}\} = \mathcal{A} \curvearrowleft \mathcal{G}$$

$$\begin{tikzcd}
T^*\mathcal{C} \arrow[r, phantom, "\cong"] \arrow[d, "\mu_\mathbb{C}"'] & \mathcal{C}\times\Omega^{1,0}(X,\mathrm{End}(E)) \arrow[r, phantom, "\cong"] & \mathcal{A}\times\Omega^{1}(X,\mathrm{End}(E,h)) \arrow[r, phantom, "\cong"] & T^*\mathcal{A} \arrow[d, "\mu_\mathbb{R}"] \\
\mathrm{Lie}\,\mathcal{G}_\mathbb{C}^* & & & \mathrm{Lie}\,\mathcal{G}^*
\end{tikzcd}$$

$\ni (\bar\partial, \Phi)$ ↖ not nec holo Higgs field

$\mapsto$ $\mathrm{Lie}\,\mathcal{G}_\mathbb{C}^* \ni 2i\bar\partial\Phi$ $\qquad$ $-F_{\bar\partial} - [\Phi,\Phi^*] \in \mathrm{Lie}\,\mathcal{G}^*$

hyperkähler ↗

$$\rightsquigarrow \quad \mu_\mathbb{C}^{-1}(0) = \{(\bar\partial,\Phi) : \Phi \text{ is } \bar\partial\text{-holo}\}$$

$$\mathcal{M}_{Hit} = \left(\mu_\mathbb{R}^{-1}\left(i\tfrac{d}{n}\mathrm{Id}\right) \cap \mu_\mathbb{C}^{-1}(0)\right)/\mathcal{G}$$

hyperkähler reduction

↖ moduli space of $\mathrm{sol}^{\mathrm{n}}$s to Hitchin's equations.

$\wr\wr$ gen. Hitchin-Kobayashi corr.

$\mathcal{H}_X^{ss}(n,d)$ moduli space of ss rk n, deg d Higgs bdles

Other features:

i) Hitchin's integrable system $h : \mathcal{H}_X^{ss}(n,d) \xrightarrow{\text{proper}} \mathcal{A}$ (take char poly of Higgs field)

ii) Non-abelian Hodge correspondence [Simpson, Corlette, Donaldson] relates $\mathcal{H}_X^{ss}(n,d)$ with moduli of connections & $GL_n$-reps of $\pi_1(X)$