§5 Counting indecomposable objects & Betti numbers

Representation theory & quiver moduli

<u>Trichotomy Thm</u> (Drozd) Over $k = \bar{k}$ a quiver is either

- finite type (only finitely many indecomp. reps $/\cong$)
- tame (indecomp. reps of a fixed dim $/\cong$ occur in finitely many 1-parameter families)

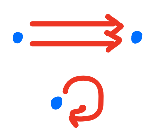
or

- wild (indecomp. reps of a fixed dim $/\cong$ occur in families with ≥ 2 parameters)

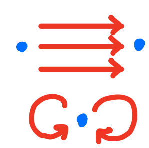
<u>Ex</u>

Finite type

Tame



Wild



<u>Thm</u> (Gabriel)

Q is of finite type $\iff$ underlying graph is a simply laced **Dynkin diagram**

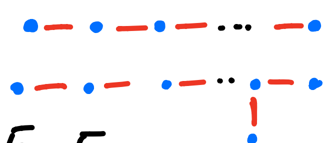
connected

$\mathfrak{g}_Q$ semisimple Lie algebra $\rightsquigarrow$

+ root system

A_n

D_n



E_6, E_7, E_8

Moreover $\left\{ \begin{matrix} \text{indecomp reps} \\ \text{of } Q \end{matrix} \Big/ \cong \right\} \xrightarrow[\dim]{\sim} \left\{ \begin{matrix} \text{positive roots} \\ \text{of } \mathfrak{g}_Q \end{matrix} \right\}$

General $Q \rightsquigarrow \mathfrak{g}_Q$ **Kac-Moody algebra**

+ root system

only depends on underlying graph of Q

<u>Thm</u> (Kac) $\left\{ \begin{matrix} \text{indecomp reps} \\ \text{of } Q \end{matrix} \Big/ \cong \right\} \xrightarrow[\dim]{} \!\!\!\!\rightarrow \left\{ \begin{matrix} \text{positive roots} \\ \text{of } \mathfrak{g}_Q \end{matrix} \right\}$

Positivity of the Kac polynomial via HK quiver moduli

Def For a finite field $\mathbb{F}_q$ define

$$A_{Q,d}(q) = \#\ \mathbb{F}_q\text{-reps of } Q \text{ of dim } d \text{ which are } \underbrace{\text{indecomp over } \overline{\mathbb{F}_q}}_{\text{absolutely indecomp reps}} \Big/ \cong$$

Thm (Kac) $A_{Q,d}(q) \in \mathbb{Z}[q]$ Kac polynomial

(Do Exercises 5.2 & 5.20)

Kac's positivity conj: The coeffs of $A_{Q,d}(q)$ are ≥ 0

Proved by Crawley-Boevey & Van den Bergh
Hausel, Rodriguez-Villegas & Letellier.

Idea: interpret coefficients as dimensions of cohomology groups of HK quiver moduli spaces for $\overline{Q}$ over $k = \mathbb{C}$

Generalisation

GIT constr & sympl constr (Gauge theory)

moduli of reps of Q $\rightsquigarrow$ moduli of vector bundles on a Riemann surface

hyperkähler version

$(k=\mathbb{C})$ moduli of reps of $\overline{Q}$ satisfying moment map rel's $\rightsquigarrow$ moduli of Higgs bundles on a Riemann surface

Many parallels: quiver reps $\leftrightsquigarrow$ vector bundles on curves

Schiffmann: Computes the dimensions of cohomology groups of Higgs moduli spaces $(k=\mathbb{C})$ by counting indecomp. vector bundles on alg. curves over finite fields.

Statement in quiver setting

Assume: Q quiver without loops

d indivisible dimension vector — i.e. d is not a multiple of another dimension vector. ($\Rightarrow$ abs. indecomp = indecomp)

θ generic stability param wrt d — i.e. θ-ss $= \theta$-s ($\forall\, d' < d$ $\sum_{v\in V}\theta_v d'_v \neq 0 = \sum_{v \in V}\theta_v d_v$)

Recall: $T^*\mathcal{M}_d^{\theta\text{-ss}}(Q) \subset \mathcal{M}_d^{\theta\text{-ss}}(\bar{Q}, R_0) = \mu^{-1}(0) /\!\!/_{\theta}\, G$

hyperkähler analogue

$\parallel$ X_0

moment map rel'ns

(hyperkähler over $k = \mathbb{C}$)

Thm (Crawley-Boevey and Van den Bergh)

For Q, d and θ as above & for a finite field $\mathbb{F}_q$ of sufficiently large characteristic

$$\underbrace{A_{Q,d}(q)}_{\substack{\text{counts of absolutely} \\ \text{indec. reps of } Q \\ \text{over } k=\mathbb{F}_q}} = \sum_{i=0}^{e} \underbrace{\dim H^{2e-2i}(X_0(\mathbb{C}), \mathbb{C})}_{\substack{\text{Betti numbers of m.space } X_0 \\ \text{of } \theta\text{-ss reps of } (\bar{Q}, R_0) \\ \text{over } k=\mathbb{C}}}\, q^i \qquad e = \tfrac{1}{2}\dim X_0$$

In particular, $A_{Q,d}(q)$ is a poly with coeffs ≥ 0

Cor: Betti no.s of the HK quiver moduli space do not depend on the stability param. θ

Outline of the proof

STEP 1:	Deforming the fibre of the moment map to produce a cohomologically trivial family

Let $X := \mathcal{M}_d^{\theta\text{-ss}}(\bar{Q}, R_{\theta}) = \mu^{-1}(\theta) /\!\!/_{\theta}\, G$

Construct a family

$$\begin{array}{ccc} \mathcal{X} & \text{with special fibre } X_0 \ \& \text{ generic fibre} & X_t \cong X_1 = X \\ \downarrow & \downarrow & \downarrow \\ \mathbb{A}^1 & 0 & t \neq 0 \end{array}$$

by setting $\mathcal{X} := \mu^{-1}(L_\theta) /\!/_\theta \, G$

where $L_\theta \subset \mathfrak{g}$ is the line joining 0 & θ

Over $k = \mathbb{C}$: Using the HK structure on $\mathrm{Rep}_{\bar{Q}}$ you can show this family is topologically trivial
$\Rightarrow$ singular cohomology of X & X_0 coincide

STEP 2: The scaling action and purity of the special fibre

We have $\mathbb{G}_m \curvearrowright \mathrm{Rep}_{\bar{Q}}$ scaling action on morphisms
with a unique fixed point $\mathrm{Rep}_{\bar{Q}}^{\mathbb{G}_m} = \{0\}$.

Moreover, $\lim_{t \to 0} t \cdot X = 0 \quad \forall \ X \in \mathrm{Rep}_{\bar{Q}}$

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathrm{Rep}_{\bar{Q}} \\ \cap & \nearrow & \\ \mathbb{A}^1 & & \end{array}$$

Def A $\mathbb{G}_m$-action on a smooth quasi-proj. variety Z is <u>semi-projective</u> if i) $\lim_{t \to 0} t \cdot z$ exists $\forall \ z \in Z$,
& ii) $Z^{\mathbb{G}_m}$ is projective.

Ex:
- $\mathbb{G}_m \curvearrowright \mathrm{Rep}_{\bar{Q}}$ is semi-projective
- Any $\mathbb{G}_m \curvearrowright \underset{\text{smooth proj}}{Z}$ is semi-projective.
- $\mathbb{G}_m \curvearrowright \mathcal{H}_X^{ss}(n,d) \leftarrow$ Higgs moduli space for $(n,d) = 1$
  $t \cdot [E, \Phi] = [E, t\Phi]$ is semi-projective

Recall $X_0 = \mu^{-1}(0) /\!\!/_{\theta} G \xrightarrow[\text{proj}]{p} \mu^{-1}(0) /\!\!/ G$ ← affine GIT quotient

$\| $

$\mathrm{Spec}\left(\mathcal{O}(\mu^{-1}(0))^G\right)$

<u>Lemma</u>:

i) The $\mathbb{G}_m$-action and G-action on $\mathrm{Rep}_{\bar{Q}}$ commute,

ii) $\mu : \mathrm{Rep}_{\bar{Q}} \longrightarrow \mathfrak{g}$ is $\mathbb{G}_m$-equiv for $\mathbb{G}_m \curvearrowright \mathfrak{g}$ weight 2,

iii) There is an induced $\mathbb{G}_m$-action on X_0 and $\mu^{-1}(0)/\!\!/G$ s.t p is $\mathbb{G}_m$-equivariant,

iv) $\mathbb{G}_m \curvearrowright \mu^{-1}(0)/\!\!/G$ is semi-projective ($\exists$ single $\mathbb{G}_m$-fixed point)

v) $\mathbb{G}_m \curvearrowright X_0$ is semi-projective.

<u>Pf</u>: Exercise.

<u>Bialynicki-Birula decomposition</u>: by considering the flow under the semi-projective $\mathbb{G}_m$-action, we obtain a deformation retract to the $\mathbb{G}_m$-fixed locus.

⇝ Cohomology of X_0 can be expressed in terms of $X_0^{\mathbb{G}_m}$ smooth proj

⇝ Cohomology of X_0 is "pure"

ie. it behaves like that of a smooth proj. variety.

STEP 3: Purity and point counting over finite fields

The Weil conjectures (+ comparison thm for sing & étale coh) enable one to compute the Betti no.s of a smooth proj var $Y/\mathbb{C}$ with good red. Z mod p by counting $\mathbb{F}_{p^r}$-points of Z.

Although X_0 is not projective, it is <u>pure</u> & we can still use finite point counts to determine the Betti nos of X_0 provided the finite point counts are <u>polynomials</u>:

<u>Lemma</u> (CB-VdB)
Let $Z/\mathbb{F}_q$ be a smooth variety which is pure and has polynomial point count: $\exists\, P(t) \in \mathbb{Z}[t]$ s.t $|Z(\mathbb{F}_{q^r})| = P(q^r)$.
Then $\sum_{i \geq 0} \dim H_c^{2i}(\bar{Z}, \mathbb{Q}_\ell) q^i = P(q)$ (ℓ coprime to p)

compactly supported ℓ-adic Poincaré polynomial

<u>Q</u>: Does X_0 have polynomial point count over finite fields?
What about the generic fibre X of $\mathfrak{X} \to \mathbb{A}^1$?

STEP 4: Point count for the generic fibre X and absolutely indecomposable vector bundles

<u>Idea</u>: relate the point count $|X(\mathbb{F}_q)|$ with the count $A_{Q,d}(q)$ of abs. indecomp. $\mathbb{F}_q$-reps of Q. when $q = p^r$ and $p \gg 0$.

Kac's Thm: polynomial

The relationship comes from work of Crawley-Boevey studying the forgetful map

$$f: \mu^{-1}(\theta) \hookrightarrow \mathrm{Rep}_{\bar{Q}} \longrightarrow \mathrm{Rep}_Q$$

<u>Thm</u> (CB): This map f has image on $\mathbb{F}_q$-points

$$\mathrm{Rep}_Q^{\mathrm{ind}}(\mathbb{F}_q) \subset \mathrm{Rep}_Q(\mathbb{F}_q)$$ set of indecomp $\mathbb{F}_q$-reps of Q

The fibre over an indecomp $\mathbb{F}_q$-rep W of Q is $\mathrm{Ext}^1_Q(W,W)^*$.

<u>Lemma</u>: Over $k=\mathbb{F}_q$ of suff. large prime characteristic

$$\mu^{-1}(\theta) = \mu^{-1}(\theta)^{\theta\text{-ss}}$$ ie the notion of θ-ss is trivial

<u>Pf</u>: It suffices to check this on $\bar{k}$ and then we can check the statement on closed points.

Note $\mu^{-1}(\theta) = \mathrm{Rep}(\bar{Q}, R_\theta)$ ← reps of $\bar{Q}$ satisfying rel^{n}s R_θ

We claim that any d-dim^e rep. of $(\bar{Q}, R_\theta)$ has <u>no</u> <u>subrepresentations</u> (so θ-ss is trivial).

If a d'-dim^e rep of $(\bar{Q}, R_\theta)$ exists, then $\sum_{v\in V} \theta_v d'_v = 0$.

(Ex: check this by taking traces of the rel^{n}s in R_θ)

However as θ is generic w.r.t. d, $\forall\ d' < d$ we have

$$\sum_{v\in V} \theta_v d'_v \neq 0,$$ which also hold in $\bar{k}$ if $\mathrm{char}(k) >> 0$.

Hence no such subrepresentation exists. ☒

<u>Thm</u> (CB-VdB) For $p >> 0$ and $q = p^r$

$$A_{Q,d}(q) = q^{-e} | X(\mathbb{F}_q)| \qquad e := \tfrac{1}{2}\dim X$$

<u>Pf</u>: Recall θ-ss on $\mu^{-1}(\theta)$ is trivial, thus

$\mu^{-1}(\theta) \twoheadrightarrow X = \mu^{-1}(\theta) /\!/_\theta G$ is a principal $\bar{G}$-bdle where $\overline{G} := G/\Delta$ for $\mathbb{G}_m \cong \Delta < G$ diagonal subgp acting trivially.

Hence $$|X(\mathbb{F}_q)| = \frac{|\mu^{-1}(\theta)(\mathbb{F}_q)|}{|\overline{G}(\mathbb{F}_q)|} \qquad (*)$$

⚠ In general over a non-alg closed field k the rat^{e} pts of the GIT quotient $X \neq$ rat^{e} orbits (even when ss = s)

eg for $k = \mathbb{R}$, X also has rat^{e} pts corr. to quaternionic quiver reps. (joint work w/ F. Schaffhauser)

In this case, as $Br(\mathbb{F}_q) = 0$: $\mathbb{F}_q$-pts of quotient = $\mathbb{F}_q$-orbits

For d indivisible dim vector : abs. indecomp. $\Leftrightarrow$ indecomp.

$$A_{Q,d}(q) = |Rep_Q^{ind}(\mathbb{F}_q) / \overline{G}(\mathbb{F}_q)|$$

Burnside's formula

$$= \frac{1}{|\overline{G}(\mathbb{F}_q)|} \sum_{W \in Rep_Q^{ind}(\mathbb{F}_q)} \underbrace{q^{-1} |End_Q(W)|}$$

$\hookrightarrow Stab_{\overline{G}}(W) \cong Aut_Q(W)/\mathbb{G}_m$ & descr. of $End_Q(W)$

Thm (CB) about the forgetful map f

$$= \frac{1}{|\overline{G}(\mathbb{F}_q)|} \sum_{\tilde{W} \in \mu^{-1}(\theta)(\mathbb{F}_q)} q^{-1} \frac{|End_Q(f(\tilde{W}))|}{|Ext^1_Q(f(\tilde{W}), f(\tilde{W}))|}$$

$$\overset{(*)}{=} q^{\langle d,d \rangle_Q - 1} |X(\mathbb{F}_q)|$$

$d = \dim f(\tilde{W})$

$$\dim End_Q(f(\tilde{W})) - \dim Ext^1_Q(f(\tilde{W}), f(\tilde{W})) = \langle d, d \rangle_Q$$ Euler form

$$e = \frac{1}{2} \dim X = \dim \mathcal{M}_d^{\theta\text{-ss}}(Q) \overset{\theta\text{-generic}}{\underset{(ss=s)}{=}} \dim Rep_Q^{\theta\text{-ss}} - \dim \overline{G} = 1 - \langle d,d \rangle_Q$$

X is alg. sympl. analogue of $\mathcal{M}_d^{\theta\text{-ss}}(Q)$

(Ex: $\langle d, d \rangle_Q = \dim G - \dim Rep_Q$) ∎

STEP 5: Kac's Thm on absolutely indecomp. reps

Thm (Kac) $A_{Q,d}(q)$ is polynomial in q.

↳ Key ideas:

- By Galois decent: suffices to show # indecomp reps $/\cong$ is polynomial
- Krull-Schmidt Thm + induction on d: suffices to show # reps $/\cong$ is polynomial
- Apply Burnside's formula + enumerate all possible Jordan normal forms ...

Cor $|X(\mathbb{F}_q)|$ is polynomial in q.

STEP 6: Specialisation and relating the cohomology of the special fibre and generic fibre

Note: Everything is defined over $\mathbb{Z}$: Rep_Q, $\mathrm{Rep}_{\bar{Q}}$ and G as well as μ, X, X_0 and $\mathfrak{X} \to \mathbb{A}^1$.

Lemma: There exists a non-empty open set $U \subset \mathrm{Spec}\,\mathbb{Z}$ over which $\mathfrak{X} \to \mathbb{A}^1$ is smooth.

Idea: Suffices to show $\mathfrak{X} \to \mathbb{A}^1$ is smooth after base change to $\overline{\mathbb{Q}}$. As G/Δ acts freely on the θ-ss locus, μ is smooth on this locus. We have

$$\begin{tikzcd} \mu^{-1}(L_\theta)^{\theta\text{-}s} \arrow[r, two heads, "(*)"] \arrow[dr, "\mu"', "(*)"] & \mathfrak{X} \arrow[d] \\ & L_\theta \cong \mathbb{A}^1 \end{tikzcd}$$

(*) both smooth $\Rightarrow$ $\mathfrak{X} \to \mathbb{A}^1$ smooth.

Proposition: For a finite field $\mathbb{F}_q$ of suff large char p, we have
$$|X_0(\mathbb{F}_q)| = |X(\mathbb{F}_q)|$$

Pf: Since $\mathfrak{X} \to \mathbb{A}^1$ is topologically trivial over $k = \mathbb{C}$ via the Comparison thm (between singular & ℓ-adic coh) + base change we obtain: for $p \gg 0$ and $\ell \neq p$
$$H^i_c(X \times_{\mathbb{F}_q} \overline{\mathbb{F}_p}, \mathbb{Q}_\ell) \cong H^i_c(X_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}_p}, \mathbb{Q}_\ell)$$
compatible with Frobenius. We've just seen that X & X_0 are both smooth if $p \gg 0$ and the claim follows from the Grothendieck-Lefschetz trace formula. $\boxtimes$

Cor: For a finite field $\mathbb{F}_q$ of suff large char. p,
$$|X_0(\mathbb{F}_q)| = q^e A_{Q,d}(q) \leftarrow \text{polynomial in } q$$

Rmk: The fact that many invariants of X & X_0 agree is encapsulated best by a motivic statement concerning $\mathbb{G}_m$-equiv. specialisations of smooth semi-projective varieties.

Thm (Hoskins-Pepin Lehalleur)
Let $\mathfrak{X}$ be a smooth q-proj. k-var. with $\mathbb{G}_m \curvearrowright \mathfrak{X}$ semi-projective
and $\mathfrak{X} \to \mathbb{A}^1$ smooth $\mathbb{G}_m$-equiv. morphism $\left(\begin{array}{c}\mathbb{G}_m \curvearrowright \mathbb{A}^1 \\ \text{positive wt}\end{array}\right)$
Then $\forall\ t \in \mathbb{A}^1$: $M(\mathfrak{X}_t) \simeq M(\mathfrak{X})$ (equality of Voevodsky motives)

Cor: Motivic non-abelian Hodge thm for Higgs moduli spaces.

To pass between GIT quotients over $k=\mathbb{C}$ and $k=\mathbb{F}_q$, we use a result about GIT over $\mathbb{Z}$ and base change.

Consider $S=\operatorname{Spec}\mathbb{Z}[1/N]$ and a variety $\mathcal{Z}$ over S

1) $\mathbb{Z}[1/N]\hookrightarrow\mathbb{C}$ gives a cx variety $\mathcal{Z}_{\mathbb{C}}=\mathcal{Z}\times_S\mathbb{C}$

2) If $p\nmid N$, $\mathbb{Z}[1/N]\twoheadrightarrow\mathbb{F}_p$ gives $\mathbb{F}_p$-var $\mathcal{Z}_{\mathbb{F}_p}$ reduction of $\mathcal{Z}$ mod p

<u>Lemma</u>: For $S=\operatorname{Spec}\mathbb{Z}[\frac{1}{N}]$ & a reductive grp scheme $\mathcal{G}/S$ acting linearly on a quasi-projective scheme $\mathcal{Z}/S$, $\exists\, \mathcal{U}\subset S$ non-empty open set over which the formation of the GIT ss locus and GIT quotient commutes with base change:

<u>ie</u> $\forall$ $\operatorname{Spec}k\longrightarrow\mathcal{U}$

$$\mathcal{Z}^{ss}\times_S k=(\mathcal{Z}\times_S k)^{ss} \quad\text{and}\quad (\mathcal{Z}/\!/\mathcal{G})\times_S k\cong(\mathcal{Z}\times_S k)/\!/\mathcal{G}\times_S k$$

<u>Pf</u> [CB-VdB, Appendix B].

<u>Note</u>: $\underset{\text{open}}{\mathcal{U}\subset} S=\operatorname{Spec}\mathbb{Z}[\frac{1}{N}]$ is of the form $\mathcal{U}=\operatorname{Spec}\mathbb{Z}[\frac{1}{M}]$ for $N|M$

$\rightsquigarrow$ can just replace N by a suff large multiple.

Then for $p\nmid N$ we have base changes

$$\begin{tikzcd}
\mathcal{Z}_{\overline{\mathbb{F}}_p} \arrow[r] \arrow[d] & \mathcal{Z}_{\mathbb{F}_{p^r}} \arrow[r] \arrow[d] & \mathcal{Z} \arrow[d] & \mathcal{Z}_{\mathbb{C}} \arrow[l] \arrow[d] \\
\operatorname{Spec}\overline{\mathbb{F}}_p \arrow[r] & \operatorname{Spec}\mathbb{F}_{p^r} \arrow[r] & S & \operatorname{Spec}\mathbb{C} \arrow[l]
\end{tikzcd}$$

which all commute with the formation of GIT quotients.

Pf of the Thm: Let $\mathbb{F}_q$ be of suff large char $p \gg 0$

s.t i) $\mathfrak{X} \to \mathbb{A}^1$ is smooth over $\mathbb{F}_q$

ii) GIT commutes with base change as above for $\mathbb{F}_q$ & $\mathbb{C}$

iii) $|X_0(\mathbb{F}_q)| = |X(\mathbb{F}_q)|$

Then X_0 is pure with polynomial point count

$$|X_0(\mathbb{F}_q)| = q^e A_{Q,d}(q)$$

$\Rightarrow$ compactly supported ℓ-adic Poincaré poly. of X_0 is given by

$$A_{Q,d}(q) = q^{-e} \sum_{i \geq 0} \dim H_c^{2i}(\overline{X}_0, \mathbb{Q}_\ell) q^i \qquad (\ell \neq p)$$

Since GIT commutes with base change (as $p \gg 0$) we have a cx variety $X_{0,\mathbb{C}}$ defined over $\mathbb{Q}$ whose reduction mod p is a $\mathbb{F}_p$-variety X_0.

Smooth base change + comparison thm $\Rightarrow$

$$A_{Q,d}(q) = q^{-e} \sum_{i \geq 0} \dim H_c^{2i}(X_{0,\mathbb{C}}, \mathbb{C}) q^i$$

Poincaré duality
$X_{0,\mathbb{C}}$ smooth of dim $2e$

$$= \sum_{i=0}^{e} \dim H^{2e-2i}(X_{0,\mathbb{C}}, \mathbb{C}) q^i$$

singular cohom of analytic var. $X_{0,\mathbb{C}}(\mathbb{C})$