§5 Counting indecomposable objects & Betti numbers

**Representation theory & quiver moduli**

**Trichotomy Thm (Drozd)** Over \( k = \overline{k} \) a quiver is either

- **finite type** (only finitely many indecomp. reps \( \sim \))
- **tame** (indecomp. reps of a fixed \( \dim \) \( \sim \) occur in finitely many 1-parameter families)
- **wild** (indecomp. reps of a fixed \( \dim \) \( \sim \) occur in families with \( \geq 2 \) parameters)

**Ex**

- \( \text{Finite type} \)
- \( \text{Tame} \)
- \( \text{Wild} \)

**Thm (Gabriel)**

\( Q \) is of finite type if and only if its underlying graph is a simply laced Dynkin diagram.

\( g_Q \) semisimple Lie algebra + root system

Moreover \( \left\{ \text{indecomp reps of } Q / \sim \right\} \xrightarrow{\dim} \left\{ \text{positive roots of } g_Q \right\} \)

**General** \( Q \mapsto g_Q \) Kac-Moody algebra + root system

**Thm (Kac)** \( \left\{ \text{indecomp reps of } Q / \sim \right\} \xrightarrow{\dim} \left\{ \text{positive roots of } g_Q \right\} \) only depends on underlying graph of \( Q \).
Positivity of the Kac polynomial via HK quiver moduli

Def: For a finite field \( \mathbb{F}_q \), define

\[ A_{Q,d}(q) = \# \text{ \( \mathbb{F}_q \)-reps of } Q \text{ of dim } d \]

which are indecomp over \( \overline{\mathbb{F}_q} \), \( \sim \)

[absolutely indecomp reps]

Thm (Kac) \( A_{Q,d}(q) \in \mathbb{Z}[q] \) \( \Leftrightarrow \) Kac polynomial

(Do Exercises 5.2, 5.20)

Kac's positivity conj: The coeff of \( A_{Q,d}(q) \) are \( \geq 0 \)

Proved by Crawley-Boevey & Van den Bergh

Hauser, Rodriguez-Villegas & Letellier.

Idea: interpret coefficients as dimensions of cohomology groups of HK quiver moduli spaces for \( \overline{Q} \) over \( k = \mathbb{C} \)

Generalisation

\( \text{moduli of reps of } Q \sim \text{moduli of vector bundles on a Riemann surface} \)

\( \text{hyperkähler version} \)

\( (k = \mathbb{C}) \text{ moduli of reps of } \overline{Q} \sim \text{moduli of Higgs bundles on a Riemann surface} \)

\( \text{GIT constr & symple constr} \)

\( \text{(Gauge Theory)} \)

Many parallels: quiver reps \( \sim \) vector bundles on curves

Schiffmann: Computes the dimensions of cohomology groups of Higgs moduli spaces \( (k = \mathbb{C}) \) by counting indecomp. vector bundles on alg. curves over finite fields.
**Statement in quiver setting**

Assume: $Q$ quiver without loops

- $d$ indivisible dimension vector
- $\Theta$ generic stability param wrt $d$

\[ T^* M_d^\Theta \subseteq M_d^\Theta \llcorner (\overline{Q}, R_0) = \mu^{-1}(0) \sslash G \]

Recall: $T^* M_d^\Theta \subseteq M_d^\Theta \llcorner (\overline{Q}, R_0) = \mu^{-1}(0) \sslash G$

Thm (Crowley-Boevey and Van den Bergh)

For $Q, d$ and $\Theta$ as above & for a finite field $FF_q$

of sufficiently large characteristic

\[ N_{Q,d}^\Theta(q) = \sum_{i=0}^{e} \dim H^{2e-2i}(X_0, C, C) q^i \quad e = \frac{1}{2} \dim X_0 \]

counts of absolutely indec. reps of $Q$

Betti numbers of $m.$ space $X_0$

over $k = FF_q$

of $\Theta$-ss reps of $(\overline{Q}, R_0)$

over $k = C$

In particular, $N_{Q,d}^\Theta(q)$ is a poly with coeffs $\geq 0$

Cor: Betti no.s of the HK quiver moduli space do not depend on the stability param. $\Theta$

**Outline of the proof**

**STEP 1:** Deforming the fibre of the moment map to produce a cohomologically trivial family

\[ X := M_d^\Theta \llcorner (\overline{Q}, R_0) = \mu^{-1}(0) \sslash G \]
Construct a family

\[ \mathcal{X} \] with special fibre \( X_0 \) and generic fibre \( X_t \cong X_1 = X \)

\[ \downarrow \quad \downarrow \quad \downarrow \]

\[ A' \quad 0 \quad t \neq 0 \]

by setting \( \mathcal{X} := \mu^{-1}(L_{O'})// G \)

where \( L_{O'}C [O'] \) is the line joining \( O \) and \( O' \).

Over \( k = \mathbb{C} \): Using the HK structure on \( \text{Rep}_Q \) you can show this family is topologically trivial

\( \Rightarrow \) singular cohomology of \( X \) and \( X_0 \) coincide

**STEP 2:** The scaling action and purity of the special fibre

We have \( G_m \triangleleft \text{Rep}_Q \) scaling action on morphisms with a unique fixed point \( \text{Rep}_Q \)\(^G_m = \{ 0 \} \).

Moreover, \( \lim_{t \to 0} t \cdot X = 0 \) \( \forall X \in \text{Rep}_Q \)

**Def:** A \( G_m \)-action on a smooth quasi-proj. variety \( Z \) is semi-projective if

1. \( \lim_{t \to 0} t \cdot z \) exists \( \forall z \in Z \),
2. \( Z^{G_m} \) is projective.

**Ex:** \( G_m \triangleleft \text{Rep}_Q \) is semi-projective

- Any \( G_m \triangleleft Z \) smooth proj.

- \( G_m \triangleleft \mathcal{H}^{ss}_x(n,d) \) Igusa moduli space for \( (n,d) = 1 \)

\( t \cdot [E, \overline{\theta}] = [E, t\overline{\theta}] \) is semi-projective
Recall \( X_0 = \mu^{-1}(0) // G \xrightarrow{p} \mu^{-1}(0) // G \hookrightarrow \text{affine } \mathbb{G}_m \text{ quotient} \quad \text{Spec}(\mathbb{O}(\mu^{-1}(0))G) \)

**Lemma:**
1. The \( \mathbb{G}_m \)-action and \( G \)-action on \( \text{Rep}_Q \) commute,
2. \( \mu : \text{Rep}_Q \rightarrow q \) is \( \mathbb{G}_m \)-equiv for \( \mathbb{G}_m \otimes q \) weight 2,
3. There is an induced \( \mathbb{G}_m \)-action on \( X_0 \) and \( \mu^{-1}(0) // G \)
   s.t. \( p \) is \( \mathbb{G}_m \)-equivariant,
4. \( \mathbb{G}_m \otimes X_0 \) is semi-projective (exists a \( \mathbb{G}_m \)-fixed point)
5. \( \mathbb{G}_m \otimes X_0 \) is semi-projective.

**Pf:** Exercise.

**Bialynicki-Birula decomposition:** by considering the flow under the semi-projective \( \mathbb{G}_m \)-action, we obtain a deformation retract to the \( \mathbb{G}_m \)-fixed locus.

\( \Rightarrow \) Cohomology of \( X_0 \) can be expressed in terms of \( X_0^{\mathbb{G}_m} \) smooth proj

\( \Rightarrow \) Cohomology of \( X_0 \) is "pure"

\( \uparrow \) it behaves like that of a smooth proj. variety.

**STEP 3:** Purity and point counting over finite fields

The Weil conjectures enable one to compute the Beth no.s of a

\[ (+ \text{ comparison thm for smooth proj var } \mathbb{C} \text{ with good red. } \mathbb{Z} \text{ mod } p \text{ by counting } \mathbb{F}_p \text{-points of } \mathbb{Z}. \]
Although $X_0$ is not projective, it is pure $8$ we can still use finite point counts to determine the Betti nos of $X_0$ provided the finite point counts are polynomials:

**Lemma (CB-VdB)**

Let $Z/F_q$ be a smooth variety which is pure and has polynomial point count: $\exists P(t) \in \mathbb{Z}[t]$ s.t. $|Z(F_{q^r})| = P(q^r)$.

Then $\sum_{i \geq 0} \dim H^i_{\text{c}}(Z, \mathbb{Q}_l) q^i = P(q)$ ($l$ coprime to $p$)

compactly supported $l$-adic Poincaré polynomial

**Q:** Does $X_0$ have polynomial point count over finite fields? What about the generic fibre $X$ of $\mathfrak{X} \to \mathcal{M}'$?

**STEP 4:** Point count for the generic fibre $X$ and absolutely indecomposable vector bundles

Idea: relate the point count $|X(F_q)|$ with the count $\mathcal{O}(Q, d)(q)$ of abs. indecomp. $F_q$-reps of $Q$ when $q = p^r$ and $p > 70$.

Kac's Thm: polynomial

The relationship comes from work of Crawley-Boevey studying the forgetful map

$\phi: \mu^1(\theta) \hookrightarrow \text{Rep}_{\mathbb{Q}} \longrightarrow \text{Rep}_{\mathbb{Q}}$
**Thm (CB):** This map $f$ has image on $\mathbb{F}_q$-points

\[
\text{Rep}_Q(\mathbb{F}_q) \subset \text{Rep}_Q(\mathbb{F}_q) \quad \text{set of indecomp } \mathbb{F}_q\text{-reps of } Q
\]

The fibre over an indecomp $\mathbb{F}_q$-rep $W$ of $Q$ is $\text{Ext}_Q^1(W,W)^*$.  

**Lemma:** Over $k=\mathbb{F}_q$ of suff. large prime characteristic

\[
\mu^{-1}(\Theta) = \mu^{-1}(\Theta)^{\Theta-ss}
\]

i.e., the notion of $\Theta$-ss is trivial.

**Pf:** It suffices to check this on $\overline{k}$ and then we can check the statement on closed points.

Note $\mu^{-1}(\Theta) = \text{Rep}(\overline{Q}, R_\Theta)$, reps of $\overline{Q}$ satisfying rel's $R_\Theta$.

We claim that any $d$-dim $\Theta$ has no subrepresentations (so $\Theta$-ss is trivial).

If a $d'$-dim $\Theta$ exists, then $\sum_{v \in V} d'_v = 0$.

(Ex: check this by taking traces of the rel's $R_\Theta$)

However, as $\Theta$ is generic wrt. $d$, $d'$, we have $\sum_{v \in V} d'_v \neq 0$, which also hold in $\overline{k}$ if $\text{char}(k) > 0$.

Hence no such subrepresentation exists. 

**Thm (CB-VdB) For $p > 0$ and $q = p^r$**

\[
\mathcal{A}_{Q,d}(q) = q^{-e} |X(\mathbb{F}_q)| \quad e := \frac{1}{2} \dim X
\]

**Pf:** Recall $\Theta$-ss on $\mu^{-1}(\Theta)$ is trivial, thus

\[
\mu^{-1}(\Theta) \to X = \mu^{-1}(\Theta)/G \quad \text{is a principal } \overline{G}\text{-bundle}
\]
where \( \overline{G} = G / \Delta \) for \( \Delta \leq G \) acting trivially.

Hence
\[
|X(F_q)| = \left| \frac{\mu^{-1}(\Theta)(F_q)}{\overline{G}(F_q)} \right| \quad (\ast)
\]

\(^!\) In general over a non-alg closed field \( k \) the \( \Theta \)-pts
of the GIT quotient \( X \neq \text{rat}^\Theta \) orbit (even when \( S_S = S \))
\eg for \( k = \mathbb{R} \), \( X \) also has \( \Theta \)-pts corr. to quaternionic quiver reps. (joint work of
F. Schaffhauser)

In this case, as \( Br(F_q) = 0 : F_q \)-pts of quotient = \( F_q \)-orbits

For \( d \) indivisible dim vector: abs. indecomp \( \Leftrightarrow \) indecomp.

\[\text{for } \Theta \text{ QLQ, } d(q) = |\text{Rep}_{\Theta}^{\text{ind}} (F_q) / \overline{G}(F_q)|\]

\( \text{Bumside's formula}\)

\[\text{Thm(CB)} \quad \text{about the forgetful map } f\]

\[\frac{1}{|\overline{G}(F_q)|} \sum_{W \in \text{Rep}_{\Theta}^{\text{ind}}(F_q)} q^{|\text{End}_{\Theta}(W)|} \quad \overset{\text{& desc of}}{\rightarrow} \quad \frac{1}{|\overline{G}(F_q)|} \sum_{\tilde{\Theta} \in \mu^{-1}(\Theta)(F_q)} q^{|\text{End}_{\Theta}(f(\tilde{W}))|} \quad \frac{1}{|\text{Ext}_{\Theta}^{1}(f(\tilde{W}), f(\tilde{W}'))|}
\]

\(\ast\)

\[q^d \cdot d_{Q}^{-1} |X(F_q)|\]

\[\text{dim } \text{End}_{\Theta}(f(\tilde{W})) - \text{dim } \text{Ext}_{\Theta}^{1}(f(\tilde{W}), f(\tilde{W})) = \langle d, d \rangle_{Q} \quad \text{Euler form}\]

\[e = \frac{1}{2} \text{dim } X = \text{dim } M_{\Theta}^{\Theta-ss}(Q) = \text{dim } \text{Rep}_{\Theta-ss}(Q) - \text{dim } \overline{G} = 1 - \langle d, d \rangle_{Q}
\]

\(X_{\Theta} \text{ is alg asymt analogue of } M_{\Theta}^{\Theta-ss}(Q)\)

\(\langle d, d \rangle_{Q} = \text{dim } G - \text{dim } \text{Rep}_{\Theta-ss}(Q)\)
STEP 5: Kac's Thm on absolutely indecomp. reps

_Thm (Kac) _\mathcal{A}_Q,d(q) is polynomial in q._

1. Key ideas:
   - By Galois descent: suffices to show \# indecomp reps \( /\cong \) is polynomial
   - Krull-Schmidt Thm + induction on \( d \): suffices to show \# reps \( /\cong \) is polynomial
   - Apply Burnside's formula + enumerate all possible Jordan normal forms...

_Cor _\( |X(F_q)| \) is polynomial in \( q \).

STEP 6: Specialisation and relating the cohomology of the special fibre and generic fibre

_Note: _Everything is defined over \( \mathbb{Z} \): \( \text{Rep}_q, \text{Rep}_{\overline{q}} \) and \( G \), as well as \( \mu, \chi, \chi_0 \) and \( \mathcal{X} \rightarrow \mathcal{A}' \).

_Lemma: _There exists a non-empty open set \( U \subset \text{Spec} \mathbb{Z} \) over which \( \mathcal{X} \rightarrow \mathcal{A}' \) is smooth.

_Idea: _Suffices to show \( \mathcal{X} \rightarrow \mathcal{A}' \) is smooth after base change to \( \overline{\mathcal{X}} \). As \( G/\Delta \) acts freely on the \( \Theta \)-ss locus, \( \mu \) is smooth on this locus. We have \( \mu^{-1}(L_0)^{G-\mathbb{S}} \xrightarrow{(*)} \mathcal{X} \)

\( (*) \) both smooth \( \Rightarrow \mathcal{X} \rightarrow \mathcal{A}' \) smooth.
Proposition: For a finite field $\mathbb{F}_q$ of suff large char $p$, we have

$$|X_0(\mathbb{F}_q)| = |X(\mathbb{F}_q)|$$

Pf: Since $X \to \mathbb{A}^1$ is topologically trivial over $k=\mathbb{C}$ via the comparison thm (between singular & $\ell$-adic coh) + base change we obtain: for $p>>0$ and $\ell \not| p$

$$H^i_c(X_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}_p}, \mathbb{Q}_\ell) = H^i_c(X_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}_p}, \mathbb{Q}_\ell)$$

compatible with Frobenius. We’ve just seen that $X$ & $X_0$ are both smooth if $p>>0$ and the claim follows from the Grothendieck-Lefschetz trace formula. \(\blacksquare\)

Cor: For a finite field $\mathbb{F}_q$ of suff large char $p$,

$$|X_0(\mathbb{F}_q)| = q^e A_{\mathbb{Q},d}(q) + \text{polynomial in }q$$

Rmk: The fact that many invariants of $X$ & $X_0$ agree is encapsulated best by a motivic statement concerning $G_m$-equiv. specialisations of smooth semi-projective varieties.

Thm (Haskins - Papan Lefschetz):

Let $X$ be a smooth $q$-proj. $k$-var. with $G_m \not\subset X$

and $X \to \mathbb{A}^1$ smooth $G_m$-equiv. morphism ($G_m \not\subset \mathbb{A}^1$

Then $\forall t \in \mathbb{A}^1$: $M(X_t) = M(X)$ (equality of Voevodsky motives)

Cor: Motivic non-abelian Hodge thm for Higgs moduli spaces.
To pass between GIT quotients over \( k = \mathbb{C} \) and \( k = \mathbb{F}_q \), we use a result about GIT over \( \mathbb{Z} \) and base change.

Consider \( S = \text{Spec} \mathbb{Z}[\frac{1}{N}] \) and a variety \( \mathcal{X} \) over \( S \):
1. \( \mathbb{Z}[\frac{1}{N}] \hookrightarrow \mathbb{C} \) gives a complex variety \( \mathcal{X}_c = \mathcal{X} \times_S \mathbb{C} \)
2. If \( pXN \), \( \mathbb{Z}[\frac{1}{N}] \to \mathbb{F}_p \) gives \( \mathbb{F}_p \)-valued \( \mathcal{X} \) \( \mathbb{F}_p \) reduction of

Lemma: For \( S = \text{Spec} \mathbb{Z}[\frac{1}{N}] \) & a reductive gp scheme \( G/S \) acting linearly on a quasi-projective scheme \( \mathcal{X}/S, \exists U \subset S \) non-empty open set over which the formation of the GIT ss locus and GIT quotient commutes with base change:

\( \exists U \subset \text{Spec} \mathbb{Z}[\frac{1}{N}] \) is of the form \( U = \text{Spec} \mathbb{Z}[\frac{1}{N}] \) for \( N|M \) open.

\( \Rightarrow \) can just replace \( N \) by a suitably large multiple.

Then for \( pXN \) we have base changes:

\[
\begin{align*}
\mathbb{Z}_{\mathbb{F}_p} & \longrightarrow \mathbb{Z}_{\mathbb{F}_p^r} \longrightarrow \mathbb{Z} \leftarrow \mathbb{Z}_c \\
\downarrow & \hspace{1cm} \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\
\text{Spec} \mathbb{F}_p & \longrightarrow \text{Spec} \mathbb{F}_p^r \longrightarrow S \leftarrow \text{Spec} \mathbb{C}
\end{align*}
\]

which all commute with the formation of GIT quotients.
Pf of the Thm: Let $\mathbb{F}_q$ be of suff. large char $p \gg 0$

s.t. i) $X \to A'$ is smooth over $\mathbb{F}_q$

ii) GIT commutes with base change as above for $\mathbb{F}_q \otimes C$

iii) $|X_0(\mathbb{F}_q)| = |X(\mathbb{F}_q)|$

Then $X_0$ is pure with polynomial point count

$|X_0(\mathbb{F}_q)| = q^e A_{Q,d}(q)$

$\Rightarrow$ compactly supported $\ell$-adic Poincaré poly. of $X_0$ is given by

$A_{Q,d}(q) = q^{-e} \sum_{i \geq 0} \dim H_{c}^{2i}(X_0, \mathbb{Q}_\ell) q^i$  \hspace{1cm} (\ell \neq p)$

Since GIT commutes with base change (as $p \gg 0$)
we have a $\mathbb{C}$x variety $X_0, \mathbb{C}$ defined over $\mathbb{Q}$ whose
reduction mod $p$ is a $\mathbb{F}_p$-variety $X_0$.

Smooth base change & comparison thm $\Rightarrow$

$A_{Q,d}(q) = q^{-e} \sum_{i \geq 0} \dim H_{c}^{2i}(X_0, \mathbb{C}) q^i$

\hspace{1cm} Poincaré duality

\hspace{1cm} $X_0, \mathbb{C}$ smooth of dim $2e$

\hspace{1cm} $X_0, \mathbb{C}$ smooth of $\dim 2e$

\hspace{1cm} singular cohom of

\hspace{1cm} analytic var. $X_0, \mathbb{C}(\mathbb{C})$

\hspace{1cm} analytic var. $X_0, \mathbb{C}(\mathbb{C})$