Syllabus for the course "Topology of geometric structures"

Local master course on Differential Topology

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What is the course about?

You may be familiar already with several examples of *geometric structures*. For instance, in other master courses you may have encountered notions like *Riemannian metrics*, *symplectic structures*, *Poisson structures*, or *immersions*.

What all these notions have in common is that they are sections of some bundle. For instance, Riemannian metrics are sections of the bundle of symmetric bilinear tensors, symplectic structures are 2-forms, Poisson structures are bivectors, and immersions are simply maps (i.e. sections of a trivialised bundle).

You may also observe that these structures are not arbitrary sections: metrics are required to be positive definite, symplectic forms have to be non-degenerate and closed, Poisson bivectors have to be involutive with respect to the Schouten bracket, and immersions must have non-degenerate differential. It follows that we want to think of geometric structures as sections of a bundle that additionally satisfy some constraint that can be expressed in terms of derivatives. For instance, the condition of being closed for a symplectic structure ω amounts to its "first derivative" $d\omega$ being zero.

The objective of this course is to introduce the necessary language to discuss and study geometric structures in general. You do not need to be already familiar with the aforementioned examples to take the course (since we won't use any of their concrete properties).

In this course we will focus on geometric structures from a *topological viewpoint*. Namely, we will ask ourselves the following question: given a manifold M, and a class of geometric structures $\mathcal{G}$, what is the homotopy type of the space $\mathcal{G}(M)$ of geometric structures on M of class $\mathcal{G}$?

This may seem like a exotic question, but here are some concrete incarnations of it:

- How many connected components does the space of immersions from M to N have?
- Is there a non-nullhomotopic loop of symplectic structures on my favourite manifold M?
- Is the space of positive scalar curvature metrics on M contractible?
- Can I see that M and N are not diffeomorphic because $\mathcal{G}(M)$ is empty but $\mathcal{G}(N)$ is not?

You may be thinking that we cannot expect general answers to these questions, since the answer probably depends rather strongly on $\mathcal{G}$. This is absolutely correct. However (quite surprinsingly!), it turns out that there are some strategies that apply in general (or at least in many cases). These tools (transversality, holonomic approximation, convex integration, removal of singularities) are collectively known as *h-principles*; understanding them is the main goal of this course.

If you want to know more about the history and applications of these ideas (and you cannot wait for the course to start!), you may want to skip ahead and read the section "Some background", right after the syllabus.

What will we cover?

At the start of the course we will become familiar with some of the foundations of Differential Topology. Namely:

- Jet bundles. I.e. how to define "derivatives" of sections in an intrinsic manner.
- Whitney topologies. I.e. what does it mean for two sections (and their derivatives up to a given order) to be "close"?
- Partial differential relations. This formalises the idea of putting "constraints" on the derivatives.
- Examples: symplectic structures, contact structures, immersions, submer-

sions, metrics with various curvature conditions. Let me know if you want me to touch on other examples!

- The h-principle philosophy. I.e. what do we mean exactly when we say that we want to determine the homotopy type of a space of geometric structures?

At this point we will have the necessary background to present the techniques that form the backbone of the course:

- The Thom-Boardman hierarchy: classifying singularities by how complicated they are.
- Thom transversality.
- Applications of transversality: existence of immersions, embeddings, and Morse functions.
- Decomposing manifolds into pieces: triangulations and handles.
- Gromov's flexible sheaves and holonomic approximation.
- Applications of flexible sheaves to open manifolds: Classification of immersions (Smale-Hirsch), submersions (Phillips), and symplectic structures (Gromov).

If time allows, we will look into other methods like convex integration, removal of singularities, or wrinkling.

What will you learn?

The main goal is for you to learn how to apply these tools to whatever examples you may encounter. In particular, at the end of the course you should be able to:

- Work with differential relations given as complements of algebraic conditions. Use dimension counting to apply transversality and determine whether a generic section is a solution.
- Use handle decompositions and Diff-invariance to obtain refined transversality statements.
- Describe the expected singularities of a generic section (or of a small family thereof).
- Determine whether a relation is open and Diff-invariant in order to apply

Gromov's method of flexible sheaves.

What should you know already?

This course is aimed at those students that have already taken a course on Differential Geometry. As such, we expect familiarity with the following notions:

- Manifolds, smooth maps, submanifolds.
- Immersion and submersion theorems.
- Bundles, including the tangent and cotangent bundle.
- Differential forms, vector fields, flows. Lie derivative and exterior differential.

A good background on Algebraic Topology is probably helpful (particularly regarding homotopy groups), but it is not strictly necessary. We will review the necessary notions during the course.

Which books will we follow?

Transversality, together with large parts of the setup, are covered in detail in:

- M. Hirsch. *Differential Topology.* Springer-Verlag (1976).
- M. Golubitsky, V. Guillemin. *Stable Mappings and Their Singularities.* Graduate Texts in Mathematics. Springer-Verlag (1973).

The former is a more elementary text.

For h-principle:

- Y. Eliashberg, N.M. Mishachev. *Introduction to the h-principle.* American Mathematical Society (2002).
- M. Gromov. *Partial differential relations.* Springer-Verlag (1986).

The first one is the standard textbook on the topic. The second one is the "bible" (and it really is an amazing book), but it can be rather unfriendly to someone learning the ideas from scratch.

There are no lecture notes for the course. However, I do recommend that you prepare your own set of notes (as my lectures will make departures from the recommended books).

What will the structure of the course be?

The course will have lectures weekly; one hour will be dedicated to examples/exercises.

The course will be graded as follows:

- 15% from the homeworks. You may expect these to be submitted every 2-3 weeks.
- 15% from the midterm.
- 70% from the final exam.

This grading scheme may be adjusted once we get closer to the beginning of the course.

Some background

A biased history of the h-principle

To begin, some potentially confusing notation:

- We will say that a mathematical statement "is an h-principle" if it computes some topological invariant of a space $\mathcal{A}$ of geometric structures by comparing it with some other space $\mathcal{B}$ that has a more "topological nature".
- A typical statement would be that $\mathcal{A}$ and $\mathcal{B}$ are homotopy equivalent (or some weaker version, a typical one being that $\mathcal{A} \to \mathcal{B}$ is surjective in homotopy groups)[1].
- The tools you use to prove an h-principle are also called h-principles.
- The whole philosophy of tackling problems like these is also called "the h-principle".

Now, with that out of the way: h-principles are a central tool in many fields. They can be found in immersion theory, embedding theory, positive scalar curvature (and other branches of Riemannian geometry), symplectic topology, contact topology, foliation theory, complex geometry... One can even find the h-principle outside of geometry and topology: in the last 15 years convex integration has revolutionised the field of fluid dynamics, allowing mathematicians to construct low-regularity solutions with exotic behaviour for the Euler equations.

This chapter is structured in three parts[2]. In Section 2.1 we review the "classic era of

[1] All the statements to appear in this course are of this form. However, you may want to know that there are also h-principle-type statements that instead involve homology or bordism groups. Two famous ones are the homological h-principle due to Vassiliev, and the proof of the Mumford conjecture due to Madsen and Weiss.

[2] What follows is an account that is *extremely* biased by my interests and the way in which I learned the subject. Without a doubt, it is missing many important contributions by a multitude of mathematicians. I apologise for this!

h-principle", which roughly goes from the 1950 to 1980. In Section 2.2 we explain how the h-principle specialises to various concrete geometries; the results covered go from the mid 1970s to today. The reader should be warned that many modern developments may not even be mentioned, as it would be impossible to be exhaustive. In Section 2.3 I discuss how one may try to apply the h-principle to other geometries (you should give it a go with your favourite one!).

2.1 The birth of the h-principle

2.1.1 The early days in immersion theory

The first h-principle-like result, from 1937, is due to Whitney and Graustein [63]: it states that immersions $\mathbb{S}^1 \to \mathbb{R}^2$ are classified, up to homotopy, by how much they rotate. Twenty years later, Smale [57] and Hirsch [47] generalised this result, proving that immersions can be classified completely in algebraic topological terms (as long as the source is smaller than the target). The Smale-Hirsch theorem is regarded as the first "modern" h-principle.

An important remark is that the Smale-Hirsch theorem does not compute the homotopy type of the space $\mathcal{A}$ of immersions. It just says that this space is in fact homotopy equivalent to the space $\mathcal{B}$ of sections of a bundle. You may be puzzled by this: what is then the point?

The crucial observation is that immersions are not sections of a bundle, they are sections of a bundle that satisfy a differential condition. The Smale-Hirsch theorem effectively allows us to "get rid of differential conditions". This is extremely useful, since the homotopy type of $\mathcal{B}$ can be computed using tools from Algebraic Topology[3].

2.1.2 The work of J. Nash

Roughly at the same time, in the 1950s, Nash proved the existence of *isometric* embeddings between Riemannian manifolds (i.e. embeddings that additionally preserve the metric infinitesimally). His work came in two flavours: first, he proved that any Riemannian manifold can be isometrically embedded into a sufficiently large Euclidean space [55]. The method he developed, called later the Nash-Moser theorem, is an implicit function theorem in infinite dimensions. This result has found many later applications: It can be used to classify solutions of generic underdetermined partial

[3]Warning: Computing the homotopy type of $\mathcal{B}$ (and thus $\mathcal{A}$) is still highly non-trivial! For instance, the *immersion conjecture*, which computes the minimal dimension N necessary to immerse a given manifold into $\mathbb{R}^N$, was only proven by R. Cohen in 1985 [9].

differential equations (many of which arise in geometry), and to provide local forms (for instance in Poisson Geometry, Hamiltonian Dynamics, and CR-geometry).

The other result [54] due to Nash (with later refinements of Kuiper) was incredibly unexpected back in 1954. It was known already back then that a smooth flat torus could not be isometrically embedded in $\mathbb{R}^3$[4] Nonetheless, Nash produced a flat embedded torus in $\mathbb{R}^3$! The catch is that this isometric embedding is only C^1 and therefore has no well-defined notion of curvature. This is a profound idea often encountered in PDEs: smooth solutions of the PDE may have very nice properties, but these are lost when we pass to low-regularity solutions. We will come back to this below.

2.1.3 Enter M. Gromov

In a series of works in the late 60s and the early 70s, M. Gromov explained how the techniques introduced by Nash and Smale were in fact concrete instances of a more general approach to Differential Topology, which he called the *h-principle* (where the h stands for homotopy).

In 1969, Gromov developed a general scheme [36, 37], called the *method of flexible sheaves*[5], capable of generalising the results of Smale-Hirsch. Namely, it classifies, up to homotopy, the solutions of any open partial differential relation invariant under diffeomorphisms, as long as the ambient manifold is open. Many concrete examples in geometry fall under the scope of this result; one of the better known ones is the classification of symplectic and contact structures in open manifolds.

In 1971, Gromov and Eliashberg developed the method of *removal of singularities* [42, 17]. The rough idea is as follows: Suppose we want to construct a section satisfying a given differential condition. A naive approach is to take an arbitrary section and see whether we can achieve the condition by "a little deformation". Very often this is not possible, but at least we obtain a section that satisfies the condition except along a subset of measure zero (say, along a submanifold), which we call the *singularity*. This is precisely what transversality is about. Removal of singularities takes this a step further: what if we analyse the singularities and we do "large deformations" to get rid of them? Eliashberg and Gromov proved that this idea can be used to reprove Smale-Hirsch and to classify maps whose singularities are "simple".

In 1972, Gromov turned Nash's implicit function theorem into an abstract result applicable in other situations [38]. Its main applications are to the classification of solutions of generic underdetermined PDEs. We regard this result as part of the method of flexible sheaves: An open differential relation is in particular underdetermined, and can

[4]Any such embedding must be curved somewhere, and therefore the pullback metric must be curved itself.

[5]The main technique therein was later called *holonomic approximation* by Eliashberg and Mishachev.

be tackled using holonomic approximation. In order to tackle closed underdetermined conditions, we need instead the analytic input of the implicit function theorem. A concrete example, worked out in detail by Gromov in the late 90s, is the classification of submanifolds tangent to generic distributions.

In 1973, Gromov took Nash's result on C^1-isometric embeddings and turned it into a general toolbox, called *convex integration* [39]. Its advantage over the method of flexible sheaves is that it can be applied to geometric structures in *closed* manifolds. The caveat is that its range of applicability is more limited (a certain condition called *ampleness* must hold for convex integration to work). A famous application is the classification of *even-contact* and *odd-symplectic* structures, due to McDuff [52]. The Smale-Hirsch theorem can also be proven with this approach.

2.2 The h-principle in some concrete geometries

After Gromov's work, from the 1970s onwards, it was understood that classification statements (for open conditions) tend to be easier in open manifolds and it was clear that many geometric structures of interest (for instance, symplectic or contact) are not described by an ample condition, so we cannot use convex integration to classify them.

This meant that new techniques were necessary to tackle geometric structures in closed manifolds.

Many of the upcoming statements can be understood as various facets of the removal of singularities approach, but they vary so wildly in their details that we cannot say that they boil down to "the same technique".

2.2.1 Contact and Symplectic Topology

In 1970 and 1971, Lutz [50] and Martinet [51] proved that every closed 3-manifold admits a contact structure. Furthermore, they showed that one can construct a contact structure in every homotopy class of plane field. Their approach was based on *surgery*: We start with the standard contact structure ξ_{std} along $\mathbb{S}^3$. According to a result of Lickorish and Wallace, every other closed 3-manifold M can be obtained from $\mathbb{S}^3$ by doing surgery along tori (i.e. we cut a solid torus out and we glue it in again in a different way). It can then be proven that this can be done in a manner compatible with the contact structure, producing a contact structure in M[6].

[6]We do not think of the Martinet-Lutz result as an h-principle, although it contains the seeds of one. The reason is that it does not readily translate to prove a classification result. Nonetheless, surgery can indeed be regarded as an h-principle technique (not unrelated to removal of singularities).

Their result can be applied to $(\mathbb{S}^3, \xi_{\mathrm{std}})$ in the following curious manner: you cut out a torus, and you reglue it exactly in the same way, but making the contact structure make "an extra turn radially in the torus". This produces a new contact structure ξ_{OT}, which is said to be obtained from ξ_{std} by "Lutz twisting".

A question is apparent at this point: are these two structures homotopic to one another? (This is the first question one may ask towards a complete classification of contact structures). Furthermore, this process can be iterated: we cut out another torus (or the same), and we add one more turn. A first guess may be that this construction produces new structures every time that turn more and more (but it does not, see below).

Rigidity in Contact Topology

In 1983, D. Bennequin [1] proved that ξ_{std} and ξ_{OT} are not homotopic. He did so by exhibiting a legendrian knot in ξ_{OT} (i.e. an embedded curve tangent to ξ_{OT}) which does not have a counterpart in ξ_{std}. This is an inspiring general idea: we can try to study geometric structures by looking at submanifolds that interact with them in a meaningful way.

We think of Bennequin's result as the birth of the *rigid* side of Contact Topology. Namely, his result proved that the h-principle is not true for contact structures (as otherwise ξ_{std} and ξ_{OT} would be homotopic). We call this *rigidity*.

We can think of rigidity as an effort to detect the extent to which the h-principle fails. Many crucial developments in Contact Topology have gone in this direction. Some of them are: The theory of *convex hypersurfaces* (i.e. a method due to E. Giroux [30, 31] that amounts to performing Morse theory in a manner adapted to the contact structure: i.e. we present our manifold as a "movie" of surfaces, and we try to detect how the contact structure changes as we move in time), the theory of *generating functions* (a procedure going back to L. Hormander that allows us to describe tangent submanifolds in certain contact manifolds in terms of functions), or the modern theory of *microlocal sheaves* (an algebraic generalisation of the former). The theory of contact surgery, as started by Martinet and Lutz, is central to this story.

When we prove rigidity (i.e. that the h-principle fails) for a geometric structure $\mathcal{G}$, we are proving that the "topological theory" of $\mathcal{G}$ does not boil down to the usual topological theory of smooth manifolds. I.e. $\mathcal{G}$ has features that cannot be detected using Algebraic Topology solely. As differential topologists, we are excited to prove something like this!

Note that, even if we need more than pure Algebraic Topology, the resulting theory can still have a "topological" flavour (as is the case in Contact Topology). Indeed, the main question under studying is still about global properties (classification up to homotopy) and not about local behaviour.

Flexibility in Contact Topology

In 1989 Y. Eliashberg [18] proved a shocking result: if you take $(\mathbb{S}^3, \xi_{\mathrm{OT}})$ and you introduce another Lutz twist, the resulting structure is homotopic to ξ_{OT}. I.e. passing from ξ_{std} to ξ_{OT} introduces a twist, but all the additional ones are not actually doing anything[7].

Eliashberg's result in full generality says that contact structures that have a Lutz twist (we normally just say that they are *overtwisted*) satisfy the h-principle. I.e. even though there is rigidity in Contact Topology, there is a subclass of structures that behave in a *flexible* manner. Another example of this phenomenon had been found by Eliashberg in the 1970s: there is a certain class of maps, called S-immersions, that do not satisfy the h-principle, but contains a subclass that does (the S-immersions with zig-zags).

The interplay between rigidity and flexibility is one of the main driving forces in Contact Topology. In practice, this means that some researchers try to prove that h-principle type results apply and others try to prove that it does not by finding invariants obstructing the existence of homotopies.

In 2012, E. Murphy [53] proved that there is an h-principle for legendrians (i.e. n-dimensional manifolds tangent to contact structures in dimension $2n+1$). However, this h-principle is also partial: It was already known that legendrians in general do not satisfy the h-principle. The result of Murphy says that there is a subclass of loose legendrians that are indeed flexible.

Other flexibility results followed shortly. In 2014, Borman-Eliashberg-Murphy [2] proved the h-principle for contact structures in all dimensions[8]. A common feature of these results is that they rely on removal of singularities and a technique, due to Eliashberg-Mishachev, called *wrinkling* [19, 21, 20]. The idea is that wrinkling constructs solutions whose singularities are very simple (unlike transversality, which produces solutions that are arbitrarily complicated), so these are easier to remove.

Rigidity in Symplectic Topology

In 1985, Gromov [40] introduced the notion of *pseudoholomorphic curves* into Symplectic Geometry. These are curves in symplectic manifolds that satisfy (a version of)

[7] A result due to Gray states that two homotopic contact structures in closed manifolds are in fact isotopic. Combining these results it follows that within ξ_{OT} you can "find as many twists as you want".

[8] The existence of contact structures in dimension 5 had been proven already, a bit earlier, by Casals-Pancholi-Presas [6].

the Cauchy-Riemann equation. The amazing feature is that the space of such curves has extremely good properties[9], and it can be used to provide invariants of symplectic manifolds and of the Hamiltonian systems they support. This was greatly generalised later by A. Floer [26] and many others.

This construction fits within the larger framework of analytic invariants built out of elliptic PDEs. This goes back to earlier work on the Dirac operator, but the great insight in the 1980s, starting with Donaldson's work on 4-manifolds [15], is that *non-linear* elliptic PDEs can provide extremely powerful invariants. The reason is that a linear elliptic PDE in a closed manifold has a finite-dimensional vector space as its space of solutions (meaning that the only invariant that one can extract is the dimension), but a non-linear elliptic PDE has a manifold as its solution space.

These results are on the side of rigidity, not the h-principle. For instance, using pseudo-holomorphic curves one may prove that there is no h-principle for symplectic structures (for instance, the contact sphere $(\mathbb{S}^3, \xi_{\mathrm{OT}})$ cannot be filled in by a symplectic ball).

There is a enormous body of work on the applicability of pseudoholomorphic curves to the study of symplectic and contact manifolds. Other invariants (of sheaf-theoretical nature) are also playing an important role more recently.

Flexibility in Symplectic Topology

On the side of flexibility, one of the key ingredients is the subtle interplay between contact and symplectic manifolds. One often states that contact manifolds are the natural boundary condition for exact symplectic manifolds. The reason is that a exact symplectic manifold comes naturally endowed with a vector field (called the *Liouville vector field*) and any hypersurface transverse to it inherits a contact structure. This idea leads to the study of symplectic cobordisms between contact manifolds and to the construction of exact symplectic manifolds in a handle-by-handle manner.

This latter idea is what the call the theory of Weinstein manifolds (i.e. manifolds built out of symplectic handles of a particular type, called Weinstein). Eliashberg (jointly later with Cieliebak) [8] proved that there is an h-principle for such objects. As one may expect, the theory of Weinstein manifolds is closely connected to Contact Topology.

Weinstein manifolds are open and have handles up to dimension half. You should recall that Gromov already proved an h-principle for symplectic structures in open manifolds. What have we gained then? The point is that a Weinstein manifold has contact boundary, whereas the symplectic structures produced by the method of flexible sheaves are incredibly wild at infinity.

[9] In good situations, it is a closed, finite-dimensional manifold. In general, you can think of it as a manifold with singularities.

The takeaway message is that, in open manifolds, constructing geometric structures that are well-behaved at infinity is still an interesting question.

2.2.2 Foliation theory

The theory of foliations in open manifolds was tackled in 1970 by A. Haefliger [45, 46], who proved that the space of foliations is homotopy equivalent to the space of Γ-structures. One of its main ingredients is the work of Gromov on flexible sheaves. The other ingredient is the definition of Γ-structure. It turns out that, unlike in other h-principle problems, one may want to classify foliations not up to homotopy, but up to concordance (i.e. up to foliated cobordism) and Γ-structures appear naturally from this perspective.

Let us make a small digression to comment on Eliashberg's classification of overtwisted contact structures in 3-manifolds. A rough sketch is as follows: We produce a triangulation of our manifold (i.e. we subdivide it into tetrahedra). Then, we apply the method of flexible sheaves in order to produce a contact structure along the faces (since a neighbourhood of the faces is an open manifold); more precisely, we mimick by hand the method of flexible sheaves, so that the structure we produce is controlled. The last step must then deal with the interior of the tetrahedra; this is done by a surgery method based on the Lutz twist (i.e. Eliashberg's result may be regarded as the h-principle counterpart of the Martinet-Lutz theorem).

A extremely similar approach was used by W. Thurston in the period 1972-76 to study foliations in closed manifolds. His approach and Eliashberg's follow the same structure. For foliations of codimension at least 2, Thurston's result [59] provides a complete classification (i.e. there is no notion of "extra twisting"). For codimension-1, Thurston's result [60] provides only existence and, in the last step, the role of the Lutz twist is played by a certain model of foliation in the solid torus called the *Reeb component*.

After Thurston's work, a natural question was whether all foliations by surfaces in a 3-manifold are homotopic to one another (i.e. whether a complete h-principle holds). This was only resolved fairly recently, in the positive, by H. Eynard-Bontemps [25]. However, there is a distinct difference with respect to Contact Topology: foliations that are homotopic need not be isotopic. This implies that, even if there is h-principle for foliations, it is still meaningful to consider foliations without Reeb components and see whether there is an interesting theory (since the Reeb component is what allows one to prove flexibility).

Indeed, in 1983-87, D. Gabai [27, 28, 29] provided an inductive method to construct so-called *taut foliations* (which is a notion more restrictive than being Reeb-less). It turns out that these objects interact in a highly non-trivial way with the topology of the ambient manifold (including its Heegaard-Floer invariants) and thus exhibit rigidity.

Taut foliations also interact closely with non-overtwisted contact structures. This is a by-product of the *confoliation* programme [24] of Eliashberg and Thurston. Namely, they showed that (almost) every foliation by surfaces in a 3-manifold can be perturbed in a C^0-manner to produce a contact structure. Later work of Bowden and Vogel explained the extent to which this approximation process is unique. This allows us to pass invariants of one structure to the other. In a way, the fact that Thurston and Eliashberg followed the same approach to construct foliations and contact structures is no surprise, since the two objects turn out to be closely related!

2.2.3 Complex Geometry

Complex structures seem a priori extremely rigid. For instance, most of the complex manifolds you know are probably affine or projective algebraic varieties (and thus studied using Algebraic Geometry). It is thus surprising that we may be able to say anything about them from a homotopy point of view.

A remarkable fact is that smooth affine varieties (often called *Stein manifolds*) can be constructed using the h-principle. Eliashberg and Cieliebak proved that any Weinstein manifold (recall that this is a exact symplectic manifold with some extra properties) can be homotoped to be Stein [8]. We stated earlier that Weinstein manifolds themselves satisfy some form of h-principle.

This result fits within a larger body of work about flexibility of Stein manifolds. This goes back to the so-called Oka principle, which says that any smooth section of a holomorphic bundle over a Stein manifold is homotopic to a holomorphic section. This was proven by Grauert [35] in 1957 and much later generalised by Gromov [41].

What these results say is that if our manifolds are simple topologically (i.e. they are made of handles of dimension half), we have h-principle for complex structures. One may then wonder whether this can be pushed to larger handles, perhaps all the way to closed manifolds. For instance, is every almost complex manifold homotopic to a complex manifold? This would in particular prove that $\mathbb{S}^6$ is complex!

2.2.4 Riemannian Geometry

Some of the conditions we care about in Riemannian Geometry (for instance, having positive/negative scalar curvature) are given by an open differential relation of second order (since the curvature is assembled out of second derivatives of the metric). A consequence of the method of flexible sheaves is then that an open manifold always admits a metric with positive/negative curvature. The caveat is, once again, that the metric is wild at infinity (and in particular probably not complete).

As such, we may ask ourselves: What about h-principles in closed manifolds? What about h-principles in open manifolds but with reasonable behaviour at infinity? In 1983, Gromov and Lawson [43] proved that any manifold of dimension n assembled out of handles of dimension at most $n-3$ admits a well-behaved metric of positive scalar curvature. It turns out that this theory parallels the theory of Weinstein manifolds in many ways. First of all, it boils down to a handle-by-handle construction in which the handles have a very special form. Secondly, one can use analytic invariants (using minimal hypersurfaces or the solutions of the Dirac equation) to detect rigidity. There has been plenty of activity in this direction in the last few years[10].

In a separate direction, we already discussed the C^1-isometric embeddings of Nash. The result cannot be improved to produce C^2 maps, but we can wonder whether maps of some intermediate Hölder regularity can be constructed. Gromov conjectured that the optimal regularity is $C^{1,1/2}$; some recent works [10] have gone in this direction.

2.2.5 Embedding theory

We have discussed immersions plenty and observed that many different h-principle techniques apply to prove the Smale-Hirsch theorem. Embeddings are much more central to the study of manifolds (since an embedded submanifold may allow us to perform surgery or other interesting constructions) so: What about h-principles for embeddings?

A first remark is that embeddings are different in nature from immersions. Namely, they exhibit non-locality. Whereas being an immersion is something we check infinitesimally at each individual point (i.e. is the differential at each point non-degenerate?), being an embedding is checked on pairs of points (i.e. are each two points mapped to different images?) All the conditions we saw before were local[11].

The first h-principle-like result for embeddings is Whitney's embedding theorem [62, 64]. It says that any map of an n-dimensional manifold M into $\mathbb{R}^{2n+1}$ can be perturbed slightly to be an embedding. The idea is that the target is so large that most maps do not self-intersect.

The first true h-principle for embeddings [44] was proven by A. Haefliger in 1962. He computed the first homotopy groups of the space of embeddings of an n-dimensional manifold M into $\mathbb{R}^N$, as long as $n < 2N/3$. Much like the Smale-Hirsch theorem, this is not a direct computation; instead, he proved that these coincide with the homotopy groups of a space assembled out of spaces of sections over M^2.

[10] But this is not something I am very familiar with.

[11] Earlier we talked about isometric embeddings and legendrian embeddings. These are conditions that are "local relative to embeddings"; i.e. all the non-locality is about being an embedding and isometry and legendrianity are local properties.

The approach of Haefliger relies on a generalisation of the Whitney trick, which may be thought of as a removal of singularities scheme. Indeed: one starts with an arbitrary immersion and tries to get rid of self-intersections (the "singularities") by pushing the immersion across itself using a Whitney disc as a guide (i.e. an auxiliary disc that indicates how self-intersections should be resolved). An alternate proof, closer to the removal of singularities approach to the Smale-Hirsch theorem, was given later by Szücs [58].

Further generalisations of the Whitney-trick approach were later given by Dax and Hatcher-Quinn. A related approach, called *disjunction*, was used by Morlet and greatly generalised by Goodwillie [32] (with eventual contributions from Weiss [61, 34] and Klein [33], among others). The idea is that, in order to improve the range of homotopy groups computed by Haefliger's theorem, one wants to relate embeddings to spaces of sections over M^k, with k going to infinity. The intuition is that increasing k allows us to "sample" the embeddings at more points, providing more information. The rough statement is that in this manner we obtain an h-principle as long as we study embeddings of codimension at least 3.

The aforementioned results of Goodwillie-Klein-Weiss are part of modern Homotopy Theory, and fit within the greater context of the Goodwillie calculus of functors. It is apparent that tools of Homotopy Theory can be incredibly useful in the study of h-principles. Other results in this direction are [16, 49]. Nonetheless, this interaction is still underexplored.

2.2.6 Fluid Dynamics

From Homotopy Theory, we now move to PDEs. We have already said that Nash's approach to C^1-isometric embeddings is reminiscent of other PDE results in which one produces low regularity solutions that exhibit exotic behaviours (in Nash's case, due to the low regularity, there is no curvature and therefore no obstructions associated to it). This PDE train of thought then leads to ask where exactly is the regularity boundary between "normal" and "exotic" behaviour (as stated above, for this problem the conjectural boundary is at $C^{1,1/2}$).

It turns out that Nash's convex integration can be generalised to other PDEs. In the last 15 years we have seen an explosive development of these methods in the realm of fluid equations. For instance, smooth solutions of the Euler equation present conservation of energy, but one can construct low-regularity solutions that do not (e.g. the fluid may go from being perfectly still to suddenly moving without an external force).

A precursor of these results is due to Scheffer [56], who in 1993 proved that there is an L^2 solution of the planar incompressible Euler equation that is non-trivial but compactly-supported. This result was later improved, using convex integration, by C. De Lellis and L. Székelyhidi [11, 12], who proved that one could build continuous solutions in the 3-torus that have any energy profile desired (i.e. energy does whatever we want). In 2016-17, work of P. Isett [48] and then Buckmaster/De Lellis/Székelyhidi/Vicol [3] was able to push this result to its optimal regularity $C^{1/3}$.

2.3 So I heard you want to apply the h-principle

To wrap up this very incomplete and biased historical account, I thought it would be interesting to include a discussion on how one may apply all these ideas in the context of modern research[12].

2.3.1 Why the h-principle

If we look at the landscape of *all* geometric structures, it is apparent that we know very little about them from an h-principle viewpoint. Namely: the geometries listed above pretty much exhaust the list of all the geometries in which the h-principle makes an appearance[13].

Let us call your favourite geometry $\mathcal{G}$. How does the h-principle apply to $\mathcal{G}$? The most likely answers are:

- $\mathcal{G}$ is one of the geometries above.
- We do not know.
- We don't care, because it is not that interesting to classify structures in $\mathcal{G}$ up to homotopy.

For the last item, the point is that the interesting features of $\mathcal{G}$ may not be invariant up to homotopy. For instance, vector fields are interesting (this is what the field of Dynamics is about), but they form a contractible space.

A more subtle point is that there are many geometries (e.g. Poisson Geometry, Sub-Riemannian Geometry) in which there are interesting properties that are clearly not invariant up to homotopy. However, it may still be the case that there is also an interesting theory up to homotopy. For instance, the theory of foliations combines both aspects.

[12]Disclaimer: The upcoming ramblings are my personal viewpoint, which may differ rather drastically from someone else's.

[13]This is not meant to be taken literally, but it is morally true. The fact is that there is only a handful of geometries whose development has been deeply impacted by the h-principle.

2.3.2 Well-established geometries

Thus: Maybe you are really interested in foliations, symplectic/contact structures, or positive scalar curvature metrics.

From the perspective of this summary, the crucial property shared by these four geometries is that they exhibit both flexibility and rigidity. Flexibility is most intriguing when there is some rigidity present, and asking ourselves where the line between the two is, is what makes these fields interesting (among other reasons).

It may be the case that other geometries $\mathcal{G}$ also present a similar behaviour at the interface of flexibility and rigidity, but maybe we just do not know yet. A comment in this direction is that, whereas h-principle methods (flexibility) are sometimes geometry-independent[14], finding invariants (rigidity) is very geometry-specific.

2.3.3 Applying the h-principle

Alright, so somehow you believe that it is meaningful to study structures of type $\mathcal{G}$ up to homotopy. How should you go about applying the h-principle philosophy to $\mathcal{G}$?

The first step is to check whether $\mathcal{G}$ is given by an open, Diff-invariant partial differential relation. If that is the case, Gromov's method of flexible sheaves will tell you that there is an h-principle in open manifolds. If $\mathcal{G}$ is given instead by an underdetermined PDE, you may try to see whether you can apply the Nash-Moser-Hamilton-Gromov implicit function theorem.

The next question is whether you can also prove the result in closed manifolds. Your first option is to check whether the differential relation defining $\mathcal{G}$ is ample. If it is, then you are done! If not, it may still be the case that ampleness fails only by a little; you can then try to modify convex integration to deal with it [13].

If convex integration does not work, you can give removal of singularities a go. Unlike the previous two, this is more of a general heuristic and less of an actual blackbox. Many h-principles are in the vein of removal of singularities, but they often differ quite dramatically in how this is implemented. For instance, the removal of singularities proof of Smale-Hirsch is based on the idea of constructing the desired immersion one coordinate (of the target) at a time; the singularities appearing as we go along are only controlled thanks to transversality.

Other removal of singularities arguments use instead a technique called wrinkling. The idea is to produce first "almost" solutions with mild singularities, which are then

[14]This is not always true, as some of the recent h-principle breakthroughs seem to be quite specific to the structure under study (for instance, the existence of higher-dimensional contact structures). Then again, maybe there is a general argument behind with greater applicability.

removed by a surgery process. A general strategy is as follows: Given any open Diff-invariant differential relation in a closed manifold, we remove a point p, yielding an open manifold. This allows us to apply the method of flexible sheaves. The point p is now the "infinity" of the complement, so the structure produced is extremely wild at p. One can then apply flexible sheaves carefully again in order to "tame" the structure close to p. This yields an almost solution that has many (now controlled) singularities close to p; we call these *wrinkles*. In order to conclude the argument, we now need a geometry-specific argument to get rid of these singularities. This type of argument appeared first in the wrinkling saga of Eliashberg-Mishachev [19, 21, 20, 22, 23], it was then used by Murphy [53] in the classification of loose legendrians, and later by Borman-Eliashberg-Murphy [2] to prove the classification of contact structures in all dimensions. The h-principles for Engel structures [7, 4, 14, 5] also follow this idea.

That is to say: the h-principle is not so much about applying known methods to new geometries (although that can be helpful/clarifying), but rather about coming up with new tools to address geometries that were previously out of reach.

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